EE 562 Midterm Solutions

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Problem	Points	Score
1	12	
2	12	
3	16	
4	20	
5	20	
6	20	
Total	100	

Instructions and Information:

1) Print your name at the top of the page and indicate whether or not you are a DEN (off-campus) student.
2) Make sure your exam has 6 problems.
3) This is a closed book, closed notes exam. You may use one 8 ½ x 11 in. sheet of notes (front and back).
4) You may use a calculator but not a computer or a cell phone or any device with internet capability.
5) **You have 90 minutes to take this exam.**
6) Partial credit will be given but you must show your work to receive any credit.
7) **Circle or box your final answers (if numerical)**.

Problem 1. Suppose X is a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta}, & x \geq 0, \\ 0, & \text{elsewhere} \end{cases}$$

where β is a positive real number.

a. Derive the moment generating function (MGF) for X.

Solution: Let $M_X(s)$ denote the MGF. Then

$$\begin{aligned} M_X(s) &= E\left[e^{sX}\right] \\ &= \frac{1}{\beta}\int_0^\infty e^{sx}e^{-x/\beta}dx \\ &= \frac{1}{\beta}\int_0^\infty e^{(s-1/\beta)x}dx \end{aligned}$$

which becomes

$$M_X(s) = \frac{1}{1-\beta s}, \quad s < \frac{1}{\beta}.$$

b. Use your MGF to aid you in computing $E\left[X^4\right]$.

Solution: We find

$$\begin{aligned} M_X'(s) &= \frac{\beta}{(1-\beta s)^2} \\ M_X^{(2)}(s) &= \frac{2\beta^2}{(1-\beta s)^3} \\ M_X^{(3)}(s) &= \frac{6\beta^3}{(1-\beta s)^4} \\ M_X^{(4)}(s) &= \frac{24\beta^4}{(1-\beta s)^5} \end{aligned}$$

hence,

$$E\left[X^4\right] = 24\beta^4.$$

Problem 2. Let $\mathbf{W}$ be a white random vector with

$$\mu_{\mathbf{W}} = (0 \quad 0)^t, \quad \mathbf{K_W} = \mathbf{I}.$$

Let

$$\mathbf{X} = \mathbf{HW} + \mathbf{c}.$$

Find $\mathbf{c}$ and a causal matrix $\mathbf{H}$ with real and positive entries that produces

$$\mu_{\mathbf{X}} = [1 \quad 1]^t, \quad \mathbf{K_X} = \begin{bmatrix} 12 & 2 \\ 2 & 6 \end{bmatrix}.$$

Show all your steps.

Solution: Clearly,

$$\mu_{\mathbf{X}} = [1 \quad 1]^t.$$

Let us write

$$\mathbf{H} = \begin{bmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{bmatrix}.$$

Then

$$\mathbf{K_X} = \mathbf{HH}^t$$

that is,

$$\begin{bmatrix} 12 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} \\ 0 & h_{22} \end{bmatrix}.$$

So,

$$12 = h_{11}^2 \Rightarrow h_{11} = \sqrt{12}$$

$$2 = h_{11}h_{21} \Rightarrow h_{21} = \frac{1}{\sqrt{3}}$$

$$6 = h_{21}^2 + h_{22}^2 \Rightarrow h_{22} = \sqrt{\frac{17}{3}}.$$

Problem 3. Suppose we have an independent sequence of random variables $X_k,\ k = 1, 2, \ldots,$ where each X_k is a ± 1 Bernoulli random variable where $P(X_k = 1) = P(X_k = -1) = 1/2$. Let

$$Y_n = \sum_{k=1}^{n} \frac{X_k}{\sqrt{k!}}.$$

a. Find the variance of Y_n as $n \to \infty$.

Solution: Since the X_k random variables are independent we get

$$Var(Y_n) = \sum_{k=1}^{n} \frac{Var(X_k)}{k!}.$$

Since $E[X_k] = 0$ we have $Var(X_k) = E[X_k^2] = 1$ hence

$$Var(Y_n) = \sum_{k=1}^{n} \frac{1}{k!}$$

or

$$Var(Y_n) = e - 1.$$

b. Show that Y_n/n converges to 0 in probability.

Solution: We will use Chebyshev's inequality to obtain

$$\begin{aligned} P\left(|Y_n/n| > \epsilon\right) &\leq \frac{E[Y_n^2/n^2]}{\epsilon^2} \\ &= \frac{1}{n^2\epsilon^2} \end{aligned}$$

so clearly $P\left(|Y_n/n| > \epsilon\right) \to 0$ as $n \to \infty$ so Y_n/n converges to 0 in probability.

Problem 4. We are given an observation of X and we must decide between two hypotheses:

$$H_0: \ X \sim f_1(x)$$
$$H_1: \ X \sim f_2(x)$$

where, $f_1(x)$ is a density of the form

$$f_1(x) = \begin{cases} 3e^{-3x}, & 0 \le x < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

and $f_2(x)$ is a density of the form

$$f_2(x) = \begin{cases} 3e^{-3(3-x)}, & -\infty < x \le 3, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find a threshold T (to 3 decimal places) so that the probability of type I error is 10^{-2}.

   Solution: We assume H_0 is true and we reject H_0 (so make a type I error) if the threshold T is exceeded. We thus solve

$$\int_T^\infty f_1(x)dx = 10^{-2}$$

   for T to get

$$\int_T^\infty 3e^{-3x}dx = 10^{-2}$$

   which gives

$$T = -\frac{1}{3}\ln(0.01)$$

   or

$$T = 1.535.$$

b. Using $T = 1.6$ (which is not the answer to part a) find the probability of a type II error.

Solution: We assume H_1 is true and we reject H_1 (so make a type II error) if the given threshold $T = 1.6$ is *not* exceeded. We thus compute

$$\int_{-\infty}^{T} f_2(x)dx = \int_{-\infty}^{1.6} 3e^{-3(3-x)}dx$$

which gives

$$P(\text{type II error}) = e^{-4.2}$$

or

$$P(\text{type II error}) = 0.015.$$

Problem 5. Design a correlation detector for deciding whether H_0 or H_1 is true where

$$H_i: \ \mathbf{X}(u) = \mathbf{S}_i + \mathbf{N}(u), \quad i = 0, 1$$

and

$$\mathbf{S}_i = (-1)^i \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$E[\mathbf{N}(u)] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{K_N} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Indicate the decision regions on a graph.

Solution: Our threshold is

$$T = \frac{\mathbf{S}_1^\dagger \mathbf{K_N}^{-1} \mathbf{S}_1 - \mathbf{S}_0^\dagger \mathbf{K_N}^{-1} \mathbf{S}_0}{2} = 0.$$

Also,

$$\mathbf{K_N}^{-1}(\mathbf{S}_1 - \mathbf{S}_0) = \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix}$$

Our correlation detector then becomes

$$[x_1 \ x_2] \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix} \leq 0 \text{ for } H_0, \text{ else choose } H_1.$$

This simplifies to

$$x_2 \leq x_1 \text{ for } H_0, \text{ else choose } H_1.$$

The decision boundary is a line thru the origin with a slope of 1. The H_0 region is below the line.

Problem 6. Let $\mathbf{X}(u)$ and $\mathbf{Y}(u)$ be random vectors related to each other by the equation

$$\mathbf{Y}(u) = \mathbf{G}\mathbf{X}(u)$$

where

$$\mathbf{G} = \begin{bmatrix} 6 & 3 & 1 & 0 \\ 1 & 6 & 3 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}, \quad \mu_{\mathbf{X}} = 0, \quad \mathbf{K}_{\mathbf{X}} = \mathbf{I}.$$

Compute the LMMSE estimator of $\mathbf{S}(u) = \begin{bmatrix} X(u,1) \\ X(u,3) \end{bmatrix}$ given $\mathbf{R}(u) = \begin{bmatrix} Y(u,1) \\ Y(u,3) \end{bmatrix}$.

Solution: The LMMSE estimator of $\mathbf{S}(u)$ is

$$\hat{\mathbf{S}}(u) = \mathbf{R}_{\mathbf{SR}}\mathbf{R}_{\mathbf{R}}^{-1}\mathbf{R}(u)$$

which becomes

$$\hat{\mathbf{S}}(u) = \frac{1}{2035}\begin{bmatrix} 276 & -54 \\ -8 & 267 \end{bmatrix}$$

or

$$\hat{\mathbf{S}}(u) = \begin{bmatrix} 0.1356 & -0.0265 \\ -0.0039 & 0.1312 \end{bmatrix}.$$