EE 562

Homework 11

Due Wednesday, April 26, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Show how to use the inverse transform method to generate a random variable X having density function

$$f(x) = \frac{e^x}{e-1},\ 0 \le x \le 1.$$

Solution:
For $0 \le x \le 1$, the CDF of X is

$$F(x) = \int_0^x \frac{e^y}{e-1}\,\mathrm{d}y = \frac{e^x - 1}{e-1}$$

Set $u = F(x)$ to get

$$u = \frac{e^x - 1}{e-1} \Rightarrow x = \ln(u(e-1)+1) \Rightarrow X = \ln(U(e-1)+1)$$

where U is uniformly distributed on $[0,1]$.

Problem 2. Let

$$\theta = \int_0^1 e^{x^2} dx.$$

a. Show how you can use two independent uniform random variables to estimte θ.

b. Show how you can use antithetic variables to estimte θ.

c. Show that the variance of the estimator in part (b) is less than the variance of the estimator in part (a).

Solution:

a. Let U be uniform in $(0,1)$. Then $\theta\left[e^{U^2}\right]$. So pick U_1 and U_2 independently and each uniformly in $[0,1]$. Then

$$\hat{\theta}_1 = \frac{e^{U_1^2} + e^{U_2^2}}{2}$$

b. Let U be uniform in $[0,1]$. Thus $1-U$ is also uniform in $[0,1]$. Then

$$\hat{\theta}_2 = \frac{e^{U^2} + e^{(1-U)^2}}{2}$$

c. We find

$$\begin{aligned}\mathrm{Var}\left(\hat{\theta}_2\right) &= \frac{1}{4}\mathrm{Var}\left(e^{U^2} + e^{(1-U)^2}\right) \\ &= \frac{1}{4}\left(\mathrm{Var}\left(e^{U^2}\right) + \mathrm{Var}\left(e^{(1-U)^2}\right) + 2\mathrm{Cov}\left(e^{U^2}, e^{(1-U)^2}\right)\right)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left(\hat{\theta}_1\right) &= \frac{1}{4}\mathrm{Var}\left(e^{U_1^2} + e^{U_2^2}\right) = \frac{1}{4}\left(\mathrm{Var}\left(e^{U_1^2}\right) + \mathrm{Var}\left(e^{U_1^2}\right)\right) \\ &= \frac{1}{4}\left(\mathrm{Var}\left(e^{U^2}\right) + \mathrm{Var}\left(e^{(1-U)^2}\right)\right)\end{aligned}$$

So we just need to show $\mathrm{Cov}\left(e^{U^2}, e^{(1-U)^2}\right) < 0$.

$$\begin{aligned}\mathrm{Cov}\left(e^{U^2}, e^{(1-U)^2}\right) &= \mathrm{E}\left[e^{U^2}e^{(1-U)^2}\right] - \mathrm{E}\left[e^{U^2}\right]\mathrm{E}\left[e^{(1-U)^2}\right] < 0 \\ &\Leftrightarrow \mathrm{E}\left[e^{U^2}e^{(1-U)^2}\right] < \mathrm{E}\left[e^{U^2}\right]\mathrm{E}\left[e^{(1-U)^2}\right]\end{aligned}$$

At this point you can expand the exponential terms in a Taylor series, take expected values as indicated and the result will follow.

Problem 3. Suppose we have $\mathbf{X} = X_1, X_2, \ldots, X_n$ where each X_i, $i = 1, 2, \ldots, n$ is a Poisson random variable with parameter λ. Then

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

We know that $E[X_i] = \lambda$ and $Var[X_i] = \lambda$. Let $W[\mathbf{X}]$ be an estimator of λ. Compute the Cramer-Rao lower bound for $Var\left(W[\mathbf{X}]\right)$. Also, show that the sample mean achieves the Cramer-Rao lower bound.

Solution:
The log likelihood function for the n samples of a Poisson random variable is

$$\ln l(x_1, x_2, \ldots, x_n; \lambda) = \sum_{j=1}^{n}\left(x_j \ln\lambda - \lambda - \ln x_j!\right)$$

The score for n observations of a Poisson random variable and its derivatives are given by

$$\frac{\partial}{\partial\lambda} l(k_1, k_2, \ldots, k_n; \lambda) = \frac{\sum_{j=1}^{n} k_j}{\lambda} - n$$

and

$$\frac{\partial^2}{\partial\lambda^2} l(k_1, k_2, \ldots, k_n; \lambda) = -\frac{\sum_{j=1}^{n} k_j}{\lambda^2}$$

The Fisher information is then:

$$I_n(\lambda) = E\left[\frac{\sum_{j=1}^{n} k_j}{\lambda^2}\right] = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

The lower bound for $\mathrm{Var}(W[\mathbf{X}])$ is given by

$$\frac{1}{I_n(\lambda)} = \frac{\lambda}{n}$$

From the condition on which the equality is achieved in Cramer-Rao inequality, we know that if an efficient unbiased estimator exists, it can be found using the maximum likelihood method.
For MLE condition, we have

$$\left.\frac{\partial}{\partial\lambda} l(k_1, k_2, \ldots, k_n; \lambda)\right|_{\lambda=\lambda^*} = \left(\frac{\sum_{j=1}^{n} k_j}{n} - \lambda^*\right)\frac{n}{\lambda^*} = 0$$

$$\Rightarrow \frac{\partial}{\partial\lambda} l(k_1, k_2, \ldots, k_n; \lambda) = \left\{\hat{\Theta}(\mathbf{k}) - \lambda^*\right\} k(\lambda^*) = 0$$

So the MLE $\lambda^* = \frac{\sum_{j=1}^{n} k_j}{n}$, which is the sample mean, achieves the Cramer-Rao lower bound.

Problem 4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x < \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

a. Let W_1 be an unbiased estimator for θ. Using the i.i.d. version of the Cramer-Rao inequality compute the lower bound for W_1.

b. Now let

$$Y = \max(X_1, X_2, \ldots, X_n).$$

So Y is the largest order statistic. Show that

$$W_2 = \frac{n+1}{n} Y$$

is an unbiased estimator for θ.

c. Show that the variance of the estimator in part (b) is less than the variance of the estimator in part (a), so the Cramer-Rao inequality does not apply in this case. Show (mathematically) why it is not applicable.

In general, if the range of the pdf depends on the parameter then the Cramer-Rao Theorem is not applicable.

Solution:

a. The i.i.d. version Cramer-Rao inequality is

$$\mathrm{Var}[W(X)] \geq \frac{1}{nE_\theta\left[\left(\frac{\mathrm{d}}{\mathrm{d}\theta} \ln f(X|\theta)\right)^2\right]}$$

The denominator here is

$$\begin{aligned} E_\theta\left[\left(\frac{\mathrm{d}}{\mathrm{d}\theta} \ln f(X|\theta)\right)^2\right] &= \int_0^\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \ln f(x|\theta)\right)^2 f(x|\theta)\,\mathrm{d}x \\ &= \frac{1}{\theta}\int_0^\theta \frac{1}{\theta^2}\,\mathrm{d}x \\ &= \frac{1}{\theta^2} \end{aligned}$$

So the lower bound is

$$\mathrm{Var}[W(X)] \geq \frac{\theta^2}{n}$$

b. The expectation value of W_2 is

$$E_\theta[W_2] = \frac{n+1}{n} E_\theta[Y]$$

First we need to calculate the pdf of Y. Because $Y = \max(X_1, X_2, \ldots, X_n)$

$$\begin{aligned}\Pr(Y < y) &= \Pr(X_1 < y)\Pr(X_2 < y)\ldots\Pr(X_n < y) \\ &= \left(\frac{y}{\theta}\right)^n\end{aligned}$$

Then the expectation value is

$$\begin{aligned}E_\theta[Y] &= \int_0^\theta \frac{n}{\theta^n} y^{n-1} y \,\mathrm{d}y \\ &= \frac{n}{n+1}\theta\end{aligned}$$

So we have

$$E_\theta[W_2] = \theta$$

which mean W_2 is an unbiased estimator of θ.

c. We need to calculate the variance of W_2 first.

$$\begin{aligned}\mathrm{Var}_\theta[W_2] &= \left(\frac{n+1}{n}\right)^2 \mathrm{Var}_\theta[Y] \\ &= \left(\frac{n+1}{n}\right)^2 \left[\frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2 \theta^2\right] \\ &= \frac{\theta^2}{n(n-1)}\end{aligned}$$

So

$$\mathrm{Var}_\theta[W_2] = \frac{\theta^2}{n(n-1)} < \frac{\theta^2}{n} = \mathrm{Var}[W(X)]$$

The Cramer-Rao inequality is not applicable because the regularity condition on $f(x|\theta)$ is not satisfied. That is to say, in general, we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\int T(x) f(x|\theta)\,\mathrm{d}x &= \frac{\partial}{\partial\theta}\int_0^\theta \frac{T(x)}{\theta}\,\mathrm{d}x \\ &\neq -\frac{1}{\theta^2}\int_0^\theta T(x)\,\mathrm{d}x \\ &= \int T(x)\left[\frac{\partial}{\partial\theta} f(x|\theta)\right]\end{aligned}$$

The partial derivative and the integration are not interchangeable.

Problem 5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 \le x \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

where $0 < \theta < \infty$. Find the MLE of θ and show that its variance $\to 0$ as $n \to \infty$.

Solution:

The log-likelihood function is

$$l = \sum_{j=1}^{n} \ln \theta X_j^{\theta-1} = n \ln \theta + (\theta - 1) \sum_{j=1}^{n} X_j$$

The MLE is

$$\hat{\theta} = -\frac{n}{\sum_{j=1}^{n} \ln X_j}$$

Then we can calculate the Fisher information

$$\begin{aligned} I_n(\theta) &= -\frac{\partial^2}{\partial \theta^2} l(\theta) \\ &= \frac{n}{\theta^2} \end{aligned}$$

Considering the asymptotic behavior of MLE estimation $\lim_{n\to\infty} \frac{\text{Var}\hat{\theta}}{1/I_n(\theta)} = 1$, we have

$$\lim_{n\to\infty} \text{Var}\left[\hat{\theta}\right] = \lim_{n\to\infty} \frac{1}{I_n(\theta)} = \lim_{n\to\infty} \frac{\theta^2}{n} = 0$$