EE 562

Homework 11

Due Wednesday, April 26, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Show how to use the inverse transform method to generate a random variable X having density function

$$f(x) = \frac{e^x}{e-1}, \; 0 \le x \le 1.$$

Problem 2. Let

$$\theta = \int_0^1 e^{x^2} dx.$$

a. Show how you can use two independent uniform random variables to estimte θ.

b. Show how you can use antithetic variables to estimte θ.

c. Show that the variance of the estimator in part (b) is less than the variance of the estimator in part (a).

Problem 3. Suppose we have $\mathbf{X} = X_1, X_2, \ldots, X_n$ where each X_i, $i = 1, 2, \ldots, n$ is a Poisson random variable with parameter λ. Then

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

We know that $E[X_i] = \lambda$ and $Var[X_i] = \lambda$. Let $W[\mathbf{X}]$ be an estimator of λ. Compute the Cramer-Rao lower bound for $Var\left(W[\mathbf{X}]\right)$. Also, show that the sample mean achieves the Cramer-Rao lower bound.

Problem 4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x < \theta, \\ 0, & \text{elsewhere}, \end{cases}$$

a. Let W_1 be an unbiased estimator for θ. Using the i.i.d. version of the Cramer-Rao inequality compute the lower bound for W_1.

b. Now let

$$Y = \max(X_1, X_2, \ldots, X_n).$$

So Y is the largest order statistic. Show that

$$W_2 = \frac{n+1}{n} Y$$

is an unbiased estimator for θ.

c. Show that the variance of the estimator in part (b) is less than the variance of the estimator in part (a), so the Cramer-Rao inequality does not apply in this case. Show (mathematically) why it is not applicable.

In general, if the range of the pdf depends on the parameter then the Cramer-Rao Theorem is not applicable.

Problem 5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $0 < \theta < \infty$. Find the MLE of θ and show that its variance $\to 0$ as $n \to \infty$.