EE 562

Homework 10

Due Wednesday, April 19, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Consider the mean square differential equation

$$\frac{dY(t)}{dt} + 2Y(t) = X(t)$$

for $t > 0$ subject to the initial condition $Y(0) = 0$. The input is

$$X(t) = 5\cos 2t + W(t)$$

where $W(t)$ is a white Gaussian noise process with mean zero and covariance function $K_W(\tau) = \sigma^2\delta(\tau)$. Find the covariance $K_Y(t_1, t_2)$ for $t_1, t_2 > 0$.

Solution:
First, we can take the first derivative of the original equation

$$\frac{\mathrm{d}\mu_Y(t_1)}{\mathrm{d}t_1} + 2\mu_Y(t_1) = \mu_X$$

Then, combine those two equations

$$\frac{\mathrm{d}(Y(t_1) - \mu_Y(t_1))}{\mathrm{d}t_1} + 2(Y(t_1) - \mu_Y(t_1)) = (X(t_1) - \mu_X)$$

Multiplying by $(X^*(t_2) - \mu_X^*)$ we get

$$\frac{\mathrm{d}(Y(t_1) - \mu_Y(_1))(X^*(t_2) - \mu_X^*)}{\mathrm{d}t_1} + 2(Y(t_1) - \mu_Y(t_1))(X^*(t_2) - \mu_X^*)$$
$$= (X(t_1) - \mu_X)(X^*(t_2) - \mu_X^*)$$

Taking expectation on both sides this becomes

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_{YX}(t_1, t_2) = \sigma^2\delta(t_1 - t_2)$$

with initial condition $K_{YX}(0, t_2) = 0$.
For $t_1 < t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(0, t_2) = 0$ which implies $K_{YX}(t_1, t_2) = 0$.
For $t_1 \geq t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_{YX}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$$

due to the jump of σ^2 at $t_1 = t_2$. Taking Laplace transforms we get

$$s_1 K_{YX}(s_1, t_2) + 2K_{YX}(s_1, t_2) = \sigma^2 e^{-s_1 t_2}$$

so

$$K_{YX}(s_1, t_2) = \frac{\sigma^2}{s_1 + 2} e^{-s_1 t_2}$$

Hence

$$K_{YX}(t_1, t_2) = \sigma^2 e^{-2(t_1 - t_2)}$$

for $t_1 \geq t_2$ and is zero otherwise.
Repeating the above procedure but now multiplying by $(Y^*(t_2) - \mu_Y^*(t_2))$, we get,

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) = K_{XY}(t_1, t_2)$$

with initial condition $K_Y(0, t_2) = 0$, or

$$\begin{aligned} \frac{\partial K_Y(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) &= K_{YX}(t_2, t_1) \\ &= \sigma^2 e^{-2(t_2 - t_1)} \end{aligned}$$

for $t_2 \geq t_1$ and is zero otherwise. So for $0 < t_1 \leq t_2$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) = \sigma^2 e^{-2(t_2 - t_1)}$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1, t_2) + 2K_Y(s_1, t_2) = \frac{\sigma^2 e^{-2t_2}}{s_1 - 2}$$

or

$$K_Y(s_1, t_2) = \frac{\sigma^2 e^{-2t_2}}{(s_1 - 2)(s_1 + 2)} = \frac{\sigma^2 e^{-2t_2}/4}{s_1 - 2} - \frac{\sigma^2 e^{-2t_2}/4}{s_1 + 2}$$

so

$$K_Y(t_1,t_2) = \frac{\sigma^2}{4} e^{-2t_2}\left(e^{2t_1} - e^{-2t_1}\right)$$

For $t_1 > t_2$

$$\frac{\partial K_Y(t_1,t_2)}{\partial t_1} + 2K_Y(t_1,t_2) = 0$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1,t_2) - K_Y(t_2,t_2)e^{-s_1 t_2} + 2K_Y(s_1,t_2) = 0$$

or

$$s_1 K_Y(s_1,t_2) - \frac{\sigma^2}{4}\left(1 - e^{-4t_2}\right)e^{-s_1 t_2} + 2K_Y(s_1,t_2) = 0$$

so

$$K_Y(s_1,t_2) = \frac{\sigma^2}{4} \frac{\left(1 - e^{-4t_2}\right)}{s_1 + 2} e^{-s_1 t_2}$$

So taking the inverse transform we get

$$K_Y(t_1,t_2) = \frac{\sigma^2}{4}\left(1 - e^{-4t_2}\right)e^{-2(t_1 - t_2)}$$

Now let $t_2 = t$, $t_1 = t + \tau$. Then

$$K_Y(t+\tau,t) = \begin{cases} \frac{\sigma^2}{4}\left(1 - e^{-4t}\right)e^{-2\tau} & \tau > 0 \\ \frac{\sigma^2}{4} e^{-2t}\left(e^{2(t+\tau)} - e^{-2(t+\tau)}\right) & \tau \le 0 \end{cases}$$

Problem 2. Suppose X is a Poisson random variable with parameter λ. Then

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Show that $E[X] = \lambda$ and $Var[X] = \lambda$.

Solution:
By definition, $E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^2}{k!} = e^{-\lambda}\lambda \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$. We can change the limit of summation by introducing $k - 1 = l$

$$E[X] = e^{-\lambda}\lambda \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

To find the variance, we can either calculate with direct derivation using definition, or use the Moment Generating function of Poisson RV

$$\begin{aligned} G(t) &= e^{-\lambda} e^{\lambda e^t} \\ G'(t) &= e^{-\lambda}\left(\lambda e^t e^{\lambda e^t}\right) \\ G''(t) &= \lambda e^{-\lambda}\left[\left(1+\lambda e^t\right) e^t e^{\lambda e^t}\right] \end{aligned}$$

And we have $G''(0) = \lambda + \lambda^2$. Therefore $\text{VAR} = \lambda + \lambda^2 - \lambda^2 = \lambda$.

Problem 3. Let $N(t)$ be a Poisson process with parameter λt. Find

$$E\left[(N(t) - N(s))^2\right]$$

for $t > s$.

Solution:
We know that for a Poisson process,

$$\begin{aligned} E[N(t)] &= \lambda t \\ E\left[N(t)^2\right] &= \lambda t + \lambda^2 t^2 \end{aligned}$$

Moreover, for $t > s$

$$\begin{aligned} E[N(t)N(s)] &= E[\{N(s) + (N(t) - N(s))\}N(s)] \\ &= E\left[N(s)^2\right] + E[N(t) - N(s)]E[N(s)] \\ &= \lambda s + \lambda^2 s^2 + \lambda(t-s)\lambda s \\ &= \lambda s + \lambda^2 st \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[\{N(t) - N(s)\}^2\right] &= E\left[N(t)^2\right] + E\left[N(s)^2\right] - 2E[N(t)N(s)] \\ &= \lambda t + \lambda^2 t^2 + \lambda s + \lambda^2 s^2 - 2\left[\lambda s + \lambda^2 st\right] \\ &= \lambda t - \lambda s + \lambda^2 t^2 + \lambda^2 s^2 - 2\lambda^2 st \end{aligned}$$

Problem 4. Give an example of a random process that is WSS but not ergodic in mean.

Solution:

One example: $X(t) = A$ for all t, where A is a zero-mean, unit variance random variable. $X(t)$ at different times are uncorrelated. Mean function of $X(t)$ is

$$\mu_X(t) = E[A] = 0$$

However, the time average of $X(t)$ is

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} A \, \mathrm{d}t = A$$

The time average does not always converges to 0.

Problem 5. Let $X(t)$ be a WSS random process. Show that

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_x(t_1, t_2) = -\frac{d^2}{d\tau^2} R_x(\tau).$$

Solution:

Because $X(t)$ is WSS, we have

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

We can make a change of variable $\tau = t_1 - t_2$ and use the chain rule

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2) &= \frac{\partial}{\partial t_1} \left[\frac{\partial R_X(t_1, t_2)}{\partial \tau} \frac{\partial \tau}{\partial t_2} \right] \\
&= -\frac{\partial}{\partial t_1} \frac{\partial R_X(t_1, t_2)}{\partial \tau} \\
&= -\frac{\partial}{\partial \tau} \frac{\partial R_X(t_1, t_2)}{\partial \tau} \frac{\partial t_1}{\partial \tau} \\
&= -\frac{\partial^2 R_X(t_1, t_2)}{\partial \tau^2} \\
&= -\frac{\mathrm{d}^2 R_X(\tau)}{\mathrm{d}\tau^2}
\end{aligned}$$