EE 562

Homework 9

Due Wednesday, April 12, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. The wide sense stationary process $X(t) = X(u,t)$ has correlation function

$$R_X(\tau) = \frac{1}{2}e^{-|\tau|}.$$

Let

$$\mu_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

as given in class. Compute $Var(\mu_T)$.

Solution:
From class notes we know that

$$\text{VAR}(\mu_T) = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)k_x(\tau)\,\mathrm{d}\tau$$

Let's assume $\mathrm{E}[X(t)] = 0$, hence $K_x(\tau) = R_x(\tau)$. Then

$$\text{VAR}(\mu_T) = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)\left(\frac{1}{2}e^{-|\tau|}\right)\mathrm{d}\tau = \frac{1}{4T}\left(2 + \frac{e^{-2T} - 1}{T}\right)$$

Problem 2. Consider the mean square differential equation

$$\frac{dY(t)}{dt} + 2Y(t) = X(t)$$

for $t > 0$ subject to the initial condition $Y(0) = 0$. The input is

$$X(t) = 5\cos 2t + W(t)$$

where $W(t)$ is a white Gaussian noise process with mean zero and covariance function $K_W(\tau) = \sigma^2\delta(\tau)$. Find $\mu_Y(t)$ for $t > 0$.

Solution:
Taking expectations of both sides we get

$$\frac{\partial \mu_y(t)}{\partial t} + 2\mu_y(t) = 5\cos 2t$$

with initial condition $\mu_y(0) = E[Y(0)] = 0$. Taking Laplace transform we get

$$s\mu_y(s) - \mu_y(0) + 2\mu_y(s) = \frac{5s}{s^2+4}$$

Solving the equation we get

$$\mu_y(s) = \frac{5s}{(s+2)(s^2+4)} = \frac{\frac{5}{4}s}{s^2+4} + \frac{\frac{5}{2}}{s^2+4} + \frac{-\frac{5}{4}}{s+2}$$

By taking inverse Laplace transform we get

$$\mu_y(t) = \frac{5}{4}\cos 2t + \frac{5}{4}\sin 2t - \frac{5}{4}e^{-2t} \quad t \geq 0$$

Problem 3. Let $X(t) = X(u,t)$ be a WSS random process with correlation function

$$R_X(\tau) = e^{-\tau^2}.$$

a. Show that the mean-square derivative $\frac{dX(t)}{dt}$ exists.

b. Find the correlation function of the process $\frac{dX(t)}{dt}$.

Solution:

a.
$$\frac{\partial^2 R_X(\tau)}{\partial \tau^2} = 4\tau^2 e^{-\tau^2} - 2e^{-2\tau^2}$$

which is continuous for all τ so $\frac{\partial X(t)}{\partial t}$ exists.

b.
$$R_{X'}(\tau) = -\frac{\partial^2 R_X(\tau)}{\partial \tau^2} = -\left(4\tau^2 e^{-\tau^2} - 2e^{-\tau^2}\right)$$

Problem 4. A wide sense stationary random process $X(t)$ has a covariance function

$$K_X(\tau) = 2\exp(-|\tau|).$$

Is $X(t)$ is ergodic in mean? Justify your answer.

Solution:
$X(t)$ is ergodic in mean. This can be seen by

$$\lim_{\tau\to\infty} K_X(\tau) = \lim_{\tau\to\infty} 2\exp(-|\tau|) = 0$$

Problem 5. Let $X(t)$ be a Wiener process. Recall $X(t)$ could be obtained as a limiting random walk where each step (up or down) was taken with probability $p = 1/2$. We derived

$$E[X(t)] = 0, \quad K_X(t_1, t_2) = \alpha \min(t_1, t_2)$$

where α was related to the step size and time increment. Let

$$Y(t) = \int_0^t X(v)dv.$$

a. Find the mean of $Y(t)$.

b. Find the variance of $Y(t)$.

Solution:

a.
$$\mathrm{E}[Y(t)] = \int_0^t \mathrm{E}[X(v)]\,\mathrm{d}v = 0$$

b. Since $\mathrm{E}[Y(t)] = 0$ we have

$$\begin{aligned}
\mathrm{VAR}[Y(t)] &= \mathrm{E}\big[Y^2(t)\big] \\
&= \mathrm{E}\left[\int_0^t X(v)\,\mathrm{d}v \int_0^t X(u)\,\mathrm{d}u\right] \\
&= \int_0^t \int_0^t \alpha \min(v,u)\,\mathrm{d}v\,\mathrm{d}u \\
&= \alpha \int_0^t \left[\int_0^u v\,\mathrm{d}v + \int_u^t u\,\mathrm{d}v\right] \mathrm{d}u \\
&= \alpha \int_0^t \left[ut - \frac{u^2}{2}\right] \mathrm{d}u \\
&= \alpha\left[\frac{t^3}{2} - \frac{t^3}{6}\right] \\
&= \frac{\alpha t^3}{3}
\end{aligned}$$