EE 562

Homework 8

Due Wednesday, April 5, 2017 at 6:40 p.m.

Problem 1. Stark and Woods 9.3.

Solution:

a.

$$\begin{aligned}\mu_X(t) &= \mathbb{E}[X(t)] \\ &= \sqrt{p}\mathbb{E}\Big[\sin\Big(2\pi f_0 t + B[n]\frac{\pi}{2}\Big)\Big] \\ &= \frac{\sqrt{p}}{2}\sin\Big(2\pi f_0 t + \frac{\pi}{2}\Big) + \frac{\sqrt{p}}{2}\sin\Big(2\pi f_0 t - \frac{\pi}{2}\Big) \\ &= \sqrt{p}\sin(2\pi f_0 t)\cos\Big(\frac{\pi}{2}\Big) \\ &= 0\end{aligned}$$

b. From the definition

$$K_{XX}(t_1,t_2) = p\mathbb{E}\Big[\sin\Big(2\pi f_0 t_1 + B[n_1]\frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 + B[n_2]\frac{\pi}{2}\Big)\Big]$$

There are two case, first, $t_1, t_2 \in [nT, (n+1)T]$. Then

$$\begin{aligned}K_{XX}(t_1,t_2) &= \frac{p}{2}\sin\Big(2\pi f_0 t_1 + \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 + \frac{\pi}{2}\Big) + \frac{p}{2}\sin\Big(2\pi f_0 t_1 - \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 - \frac{\pi}{2}\Big) \\ &= p\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2)\end{aligned}$$

Second, when t_1 and t_2 do not belong to the same interval, we have

$$\begin{aligned}K_{XX}(t_1,t_2) &= \frac{p}{4}\sin\Big(2\pi f_0 t_1 + \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 + \frac{\pi}{2}\Big) + \frac{p}{4}\sin\Big(2\pi f_0 t_1 + \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 - \frac{\pi}{2}\Big) \\ &+ \frac{p}{4}\sin\Big(2\pi f_0 t_1 - \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 + \frac{\pi}{2}\Big) + \frac{p}{4}\sin\Big(2\pi f_0 t_1 - \frac{\pi}{2}\Big)\sin\Big(2\pi f_0 t_2 - \frac{\pi}{2}\Big) \\ &= \frac{p}{2}\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2) - \frac{p}{2}\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2) \\ &= 0\end{aligned}$$

So, the final result is

$$K_{XX}(t_1,t_2) = \begin{cases} p\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2) & nT \le t_1, t_2 < (n+1)T \quad \text{for all n} \\ 0 & \text{else}\end{cases}$$

Problem 2. Stark and Woods 9.15.

Solution:

a. From the definition

$$\begin{aligned} R_{XX}(t_1,t_2) &= \mathbb{E}[(W_1(t_1)-W_2(t_1))(W_1(t_2)-W_2(t_2))] \\ &= R_{W_1W_1}(t_1,t_2)+R_{W_2W_2}(t_1,t_2) \\ &= (\alpha_1+\alpha_2)\min(t_1,t_2) \end{aligned}$$

for all $t_1, t_2 > 0$

b. Because Wiener process is Gaussian, $X(t)$ is a Gaussian random variable for any $t > 0$. The mean is $\mu_X = \mu_{W_1} + \mu_{W_2} = 0$. The variance is $\text{VAR}[X(t)] = R_{XX}(t,t) = (\alpha_1+\alpha_2)t$. So the pdf of $X(t)$ is

$$f_X(x;t) = \frac{1}{\sqrt{2\pi(\alpha_1+\alpha_2)t}} e^{-\frac{x^2}{2(\alpha_1+\alpha_2)t}}$$

Problem 3. Stark and Woods 10.2.

Solution:

a. We know that $R_{XX}(t,s) = K_{XX}(t,s) - \mu_X^2$. We can calculate the second order derivative

$$\begin{aligned} \left.\frac{\partial^2 R_{XX}(t,s)}{\partial t \partial s}\right|_{t=t,s=t} &= \left.\frac{\partial^2 K_{XX}(t,s)}{\partial t \partial s}\right|_{t=t,s=t} \\ &= \omega_0^2\sigma^2\cos\omega_0(t-t) \\ &= 0 \end{aligned}$$

Since $\frac{\partial^2 R_{XX}(t,s)}{\partial t \partial s}$ exist for all (t,t), the m.s. derivative $X'(t)$ exists.

b. c. First, we have

$$\mu_{X'}(t) = d\mu_X(t)/\,\mathrm{dt} = 0$$

Then

$$\begin{aligned} K_{X'X}(t,s) = R_{X'X'}(t,s) &= \frac{\partial^2 R_{XX}(t,s)}{\partial t \partial s} \\ &= \omega_0^2\sigma^2\cos\omega_0(t-s) \end{aligned}$$

Problem 4. Stark and Woods 10.4.

Solution:

a. We know that $R_{XX}(\tau) = K_{XX}(\tau) - \mu_X^2$. By taking the second derivative, we have

$$\begin{aligned}\frac{\mathrm{d}^2 R_{XX}(\tau)}{\mathrm{d}\tau^2} &= \frac{\mathrm{d}^2 K_{XX}(\tau)}{\mathrm{d}\tau^2} \\ &= \frac{2\sigma_X^2\alpha^2(-1+3\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^2}\end{aligned}$$

Since $\frac{\partial^2 R_{XX}(\tau)}{\partial\tau^2}$ exists for all τ, the m.s. derivative exists for all t.

b. First,

$$\mu_{X'}(t) = d\mu_X(t)/\,\mathrm{d}t = 0$$

And second,

$$\begin{aligned}K_{X'X'}(\tau) &= R_{X'X'}(\tau) \\ &= -\frac{\mathrm{d}^2 R_{XX}(\tau)}{\mathrm{d}\tau^2} \\ &= \frac{2\sigma_X^2\alpha^2(1-3\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^2}\end{aligned}$$

Problem 5. A random process $X(t)$, $t \in T$, is said to be continuous in probability if for every $\epsilon > 0$ and $t \in T$

$$\lim_{h\to 0} P\left\{|X(t+h) - X(t)| > \epsilon\right\} = 0.$$

Show that the Wiener process is continuous in probability.

Solution:
For Wiener process, $Y = X(t+h) - X(t)$ is a Gaussian random variable with $\mu_Y = 0$ and $\sigma_Y^2 = \alpha h$.

$$f_Y(y;h) = \frac{1}{\sqrt{2\pi\alpha h}} e^{-\frac{y^2}{2\alpha h}}$$

We also have

$$\begin{aligned} P\left\{|X(t+h) - X(t)| > \epsilon\right\} &= P\{y > \epsilon\} + P[y < -\epsilon] \\ &= 2Q\left(\frac{\epsilon}{\sqrt{\alpha h}}\right) \end{aligned}$$

So,

$$\begin{aligned} \lim_{h \to 0} P\left\{|X(t+h) - X(t)| > \epsilon\right\} &= \lim_{h \to 0} 2Q\left(\frac{\epsilon}{\sqrt{\alpha h}}\right) \\ &= 0 \end{aligned}$$

for any $\epsilon > 0$ and $t \in T$. And Wiener process is continuous in probability.