EE 562

Homework 7

Due Wednesday, March 29, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Let $X_1, X_2, \ldots, X_n$ denote an i.i.d. sequence of random variables each normally distributed with mean μ and variance σ^2 (both unknown). Derive the MLEs for μ and σ^2.

Solution:
The likelihood function of a single experiment is

$$L\big(\mu, \sigma^2, x_1, \ldots, x_n\big) = \big(2\pi\sigma^2\big)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\mu)^2\right)$$

The log-likelihood function is

$$l\big(\mu, \sigma^2, x_1, \ldots, x_n\big) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\big(\sigma^2\big) - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\mu)^2$$

The MLE should satisfy the following condition $\frac{\partial l}{\partial \hat{\mu}} = 0$ and $\frac{\partial l}{\partial \hat{\sigma^2}} = 0$. From the two conditions, we know

$$\frac{\partial}{\partial \mu}\left(\sum_{j=1}^{n}(x_j-\mu)^2\right) = 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$\frac{\partial}{\partial \sigma^2}\left(n\ln\big(\sigma^2\big) + \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\mu)^2\right) = 0 \Rightarrow \hat{\sigma^2} = \frac{1}{n}\sum_{j=1}^{n}(x_j-\mu)^2$$

Combining above expressions, we have

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{j=1}^{n}(x_j-\hat{\mu})^2$$

Problem 2. Let $X_1, X_2, \ldots, X_n$ denote an i.i.d. sequence of random variables each uniformly distributed in $[\theta,\ \theta+1]$. Find an MLE for θ.

Solution:
The likelihood function is

$$L(\theta, x_1, \ldots, x_n) = \prod_{j=1}^{n} I_{[\theta,\theta+1]}(x_i)$$

where $I_{[\theta,\theta+1]}(x_i)$ is the indicator function of $x_i \in [\theta, \theta+1]$. The log-likelihood function is

$$l(\theta, x_1, \ldots, x_n) = \sum_{j=1}^{n} \ln\bigl(I_{[\theta,\theta+1]}(x_i)\bigr)$$

Without loss of generality, let $x_1 < x_2 < \ldots < x_n$ be the ordered statistics. Then any $\hat{\theta}$ between $x_n - 1$ and x_1 is an estimate for θ since any estimate in that range maximizes the likelihood function.

Problem 3. Consider a random sequence $x(n)$ with zero mean and covariance

$$K_X(m,n) = a^{|m-n|}.$$

A second sequence $y(n)$ is generated as

$$y(n) = x(n) - x(n-1).$$

a. Compute the cross covariance of x and y and the corresponding cross spectral density.

b. Compute the covariance of y and its power spectral density.

Solution:

a. First the power spectrum of $x(n)$ is

$$S_x(f) = \frac{1-a^2}{(a-e^{-i2\pi f})(a-e^{i2\pi f})}$$

Using the formula in the lecture

$$S_{XY}(f) = H(f)^* S_x(f) = \bigl(1 - e^{i2\pi f}\bigr) S_X(f)$$

b. Similarly,

$$\begin{aligned} S_Y(f) = |H(f)|^2 S_X(f) &= \left(2 - e^{i2\pi f} - e^{-i2\pi f}\right) S_X(f) \\ &= 2(1 - \cos(2\pi f)) S_X(f) \end{aligned}$$

Problem 4. A zero mean sequence of i.i.d. random variables $x(n)$ form the input to a causal linear system defined by the linear difference equation

$$y(n) = \alpha y(n-1) + x(n), \quad |\alpha| < 1.$$

Find the impulse response $h(n)$ of a second system such that the sequence

$$z(n) = \sum_{i=0}^{k} h(i) y(n-i)$$

has a constant power spectral density, i.e.,

$$S_Z(f) = \sigma_z^2, \quad f \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Solution:
Without loss of generality, we can assume $x(n)$ has unit variance so that $S_X(f) = 1$. Start by noting that

$$y(n) = \sum_{i=0}^{\infty} \alpha^i x(n-i)$$

which means that $Y(z) = G(z)X(z)$, where

$$G(z) = \frac{1}{1 - \alpha z^{-1}}$$

We want to find $H(z)$ such that $H(z)G(z) = \sigma_z$ so that $S_Z(f) = \sigma_z^2$. We thus choose

$$H(z) = \frac{\sigma_z}{G(z)} = \sigma_z\left(1 - \alpha z^{-1}\right)$$

which gives the FIR filter:

$$h(n) = \sigma_z(\delta_K(n) - \alpha\delta_K(n-1))$$

Problem 5. Suppose that you can observe a signal which is the sum of two mutually orthogonal, WSS, zero-mean random sequences which is passed through a known linear system, i.e.,

$$y(n) = \sum_{k=0}^{N} h(k)[x_1(n-k) + x_2(n-k)].$$

Assume their covariances are known.

a. Construct a Wiener filter to optimally recover the sequence $x_1(n)$ from $y(n)$.

b. Suppose that the signal is corrupted with additive noise, i.e.,

$$z(n) = y(n) + w(n)$$

where $w(n)$ is WSS and uncorrelated with the sequences $x_1(n)$ and $x_2(n)$. Find an optimal Wiener filter to recover $x_1(n)$ from $z(n)$.

Solution:

a. Because $x_1(n)$ and $x_2(n)$ are mutually orthogonal, we have for any n and m:

$$\mathbb{E}[x_1(n)x_2(m)] = 0$$

We know the form of Wiener filter is

$$G(f) = \frac{S_{x_1y}(f)}{S_y(f)}$$

So we need to calculate $S_{x_1y}(f)$ and $S_y(f)$:

$$\begin{aligned} K_{x_1y}(m) &= \mathbb{E}[x_1(n)y(n-m)] \\ &= \mathbb{E}\left[x_1(n)\sum_{k=0}^{N} h(k)[x_1(n-m-k) + x_2(n-m-k)]\right] \\ &= \sum_{k=0}^{N} h(k)K_{x_1}(m+k) \end{aligned}$$

So we have

$$S_{x_1y}(f) = S_{x_1}(f)H^*(f)$$

For the same reason

$$\begin{aligned}
K_y(m) &= \mathbb{E}[y(n)y(n-m)] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} h(k)[x_1(n-k)+x_2(n-k)] \sum_{l=0}^{N} h(l)[x_1(n-m-l)+x_2(n-m-l)]\right] \\
&= \sum_{k=0}^{N}\sum_{l=0}^{N} h(k)h(l)K_{x_1}(l+m-k) + \sum_{k=0}^{N}\sum_{l=0}^{N} h(k)h(l)K_{x_2}(l+m-k)
\end{aligned}$$

So

$$S_y(f) = S_{x_1}(f)|H(f)|^2 + S_{x_2}(f)|H(f)|^2$$

Finally, the Wiener filter to recover $x_1(n)$ from $y(n)$ is

$$G(f) = \frac{S_{x_1y}(f)}{S_y(f)} = \frac{S_{x_1}(f)H^*(f)}{(S_{x_1}(f)+S_{x_2}(f))|H(f)|^2}$$

b. Here we need $S_{x_1z}(f)$ and $S_z(f)$. Because $w(n)$ is uncorrelated with the zero-mean sequences $x_1(n)$ and $x_2(n)$:

$$\begin{aligned}
K_{x_1z}(m) &= K_{x_1y}(m) + K_{x_1w}(m) = K_{x_1y}(m) \\
&\Rightarrow S_{x_1z}(f) = S_{x_1y}(f)
\end{aligned}$$

and

$$\begin{aligned}
K_z(m) &= K_y(m) + K_w(m) \\
&\Rightarrow S_z(f) = S_y(f) + S_w(f)
\end{aligned}$$

The Wiener filter t o recover $x_1(n)$ from $z(n)$ is thus:

$$G(f) = \frac{S_{x_1}(f)H^*(f)}{[S_{x_1}(f)+S_{x_2}(f)]|H(f)|^2 + S_w(f)}$$