# EE 562

## Homework 6

Due Wednesday, March 1, 2017 at 6:40 p.m.

**Work all 5 problems**.

**Problem 1.** Let $X_i,\ i = 1, \ldots, n$ be $n$ random variables each with mean $\mu$ and variance $\sigma^2$. Consider the sample mean defined as

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that $\hat{\mu}$ is not the MMSE of $\mu$ by finding a constant $a$ such that for any finite $n$, $a\hat{\mu}$ generates a lower MMSE of $\mu$.

**Problem 2.** Stark and Woods Problems 6.16 and 6.17. Note: The problem statements should refer to equation 6.4-2. These problems are repeated here. Let $X_1, \ldots X_n$ be $n$ independent random variables each with mean $\mu$ and variance $\sigma^2 < \infty$. Let us estimate the variance as

$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

a. Show that $\hat{\sigma^2}$ is an unbiased estimator for $\sigma^2$.

b. Show that this sample variance $\hat{\sigma^2}$ is consistent for $\sigma^2$.

**Problem 3.** Let $\mathbf{X}(u)$ and $\mathbf{Y}(u)$ be random vectors related to each other by the equation

$$\mathbf{Y}(u) = \mathbf{G}\mathbf{X}(u)$$

where

$$\mathbf{G} = \begin{bmatrix} 6 & 2 & 3 & 0 \\ 1 & 6 & 2 & 3 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}, \quad \mu_{\mathbf{X}} = 0, \quad \mathbf{K}_{\mathbf{X}} = \mathbf{I}.$$

a. Compute the LMMSE estimator of $\mathbf{S}(u) = \begin{bmatrix} X(u,3) \\ X(u,4) \end{bmatrix}$ given $\mathbf{R}(u) = \begin{bmatrix} Y(u,1) \\ Y(u,2) \end{bmatrix}$.

b. Compute the LMMSE estimator of $\mathbf{S}(u) = \begin{bmatrix} X(u,1) \\ X(u,2) \end{bmatrix}$ given $\mathbf{R}(u) = \begin{bmatrix} Y(u,3) \\ Y(u,4) \end{bmatrix}$.

c. Compute the LMMSE estimator of $\mathbf{Y}(u)$ given $\mathbf{R}(u) = \begin{bmatrix} X(u,1) \\ X(u,2) \end{bmatrix}$.

**Problem 4.** Let $X_1,\ X_2,\ X_3$ be real random variables with known means $E[X_i] = \mu_i$ and variances $\mathrm{Var}(X_i) = \sigma_i^2,\ i = 1,2,3$ and covariances $\mathrm{Cov}(X_i, X_j) = \sigma_{ij},\ i,j = 1,2,3$ for $i \neq j$ where

$$\sigma_2^2 = 1, \quad \sigma_3^2 = 2, \quad \sigma_{12} = 1/2, \quad \sigma_{13} = 4/3, \quad \sigma_{23} = 1.$$

a. Consider $\hat{X}_1$, the best linear predictor of $X_1$ given $X_2, X_3$, in the form

$$\hat{X}_1 = \alpha_2(X_2 - \mu_2) + \alpha_3(X_3 - \mu_3) + \mu_1.$$

Find the numerical values of the coefficients $\alpha_2$ and $\alpha_3$.

b. If the real vector $(X_1, X_2, X_3)$ is multivariate normal with moments as given above, find $E[X_1 | X_2, X_3]$, the conditional expectation of $X_1$ given $X_2,\ X_3$.

**Problem 5.** Consider a real vector observation of the form

$$\mathbf{X}(u) = A(u)\mathbf{S} + \mathbf{n}(u)$$

where $A(u)$ and $\mathbf{n}(u)$ are independent and the noise $\mathbf{n}(u)$ is mean zero with known covariance matrix $\mathbf{K_n}$. Also

$$P(A(u) = 1) = p, \quad P(A(u) = -1) = 1 - p.$$

You may assume that $\mathbf{K_X}$ and $\mathbf{K_n}$ are nonsingular.

a. Compute the LMMSE estimator of $A(u)$ given the observation $\mathbf{X}(u)$.

b. Compute the mean-square estimation error for your answer in part (a).