EE 562

Homework 5

Due Wednesday, February 22, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Let R_i, $i = 1, \ldots, 32$ be 32 independent random variables resulting from the envelope detection of a signal plus noise process. Assume that each R_i resulted from the envelope detection of a complex signal plus noise component.

Assuming integration detection with $N = 32$ use Albersheim's equation to plot probability of detection (P_d) vs. SNR (dB) for a probability of false alarm (P_{fa}) of 10^{-6}.

Solution:
We nned to plot SNR vs P_d using Albersheim's equation:

$$SNR = -5\log_{10} N + \left(6.2 + \frac{4.54}{\sqrt{N + 0.44}}\right) \cdot \log_{10}\left(A + 0.12AB + 1.7B\right)$$

where $A = \log\left(\frac{0.62}{P_{fa}}\right)$,$B = \log\left(\frac{P_d}{1-P_d}\right)$. Figure 1 shows the plot for P_d from 0.1 to 0.9(the range of P_d for which Albersheim's equation is a reasonable approximation).

```
clear
N=16; Pfa=0.00001;
Pd=0.1:0.05:0.9;
A=log(0.62/Pfa);B=log(Pd./(1-Pd));
SNR=-5*log10(N)+(6.2+4.54/sqrt(N+0.44))*log10(A+0.12*A.*B+1.7*B);
plot(SNR,Pd)
xlabel('SNR(db)');
ylabel('P_d')
```

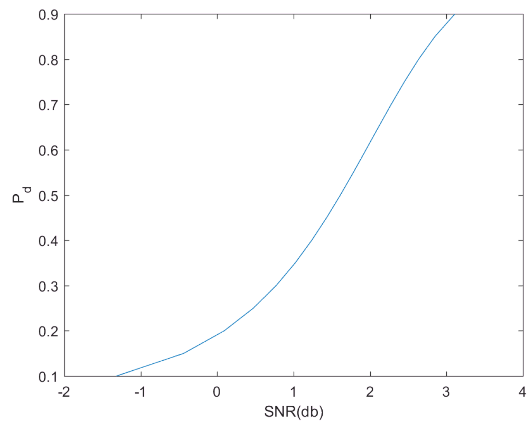

Figure 1: SNR vs Pd

Problem 2. Same setup as Problem 1. We know that the density of each R_i when no signal is present is

$$f_{R_i}(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0$$

where σ^2 is the total noise power.

With M of N detection a threshold is found for each of the 32 samples as

$$T_0 = \sqrt{-2\sigma^2 \ln(P_{fa,s})}$$

where $P_{fa,s}$ is the probability of false alarm on a sample basis that yields an overall P_{fa}.

a. Using

$$P_{fa} = \sum_{K=M}^{N} \binom{N}{K} P_{fa,s}^{K} (1 - P_{fa,s})^{N-K}$$

find $P_{fa,s}$ that yields an overall $P_{fa} = 10^{-6}$ where $M = 16$ and $N = 32$.

b. Using

$$P_d = \sum_{K=M}^{N} \binom{N}{K} P_{d,s}^K (1 - P_{d,s})^{N-K}$$

find $P_{d,s}$, the probability of detection on a sample basis, that yields an overall $P_d = 0.9$ where $M = 16$ and $N = 32$.

Solution:
In part (a) and (b), we need to invert the equations (with $M = 16$ and $N = 32$):

$$P_{fa} = \sum_{K=m}^{N} \binom{N}{K} P_{fa,s}^K (1 - P_{fa,s})^{N-K}$$
$$P_d = \sum_{K=M}^{N} \binom{N}{K} P_{d,s}^K (1 - P_{d,s})^{N-K}$$

in order to determine the $P_{fa,s}$ such that $P_{fa} = 10^{-6}$ and such that $P_d = 0.9$. Such inversion is impossible analytically. We can instead find the roots of the equations:

$$f_1(x) = 10^{-6} - \sum_{K=m} \binom{N}{K} x^K (1 - x)^{N-K}$$
$$f_2(x) = 0.9 - \sum_{K=m}^{N} \binom{N}{K} x^K (1 - x)^{N-K}$$

which can be done numerically. (For example, you can use MATLAB's fzero() function to do this.) The required $Pfa{,}s$ and $Pd{,}s$ are:

$$P(fa, s) = 0.1367$$
$$P(d, s) = 0.596$$

Problem 3. Let $\{x_n\}$ be an orthonormal set in a pre-Hilbert (or inner product space) H. Show for any x in H

$$\sum_n | < x, x_n > |^2 \leq ||x||^2.$$

Recall that $\{x_n\}$ an orthonormal set means $< x_m, x_n >= 0$ if $m \neq n$ and $< x_n, x_n >= 1$.

Solution:

$$0 \leq ||x - \sum_{i=1}^{N} < x, x_i > ||^2 =< x - \sum_{i=1}^{N} < x, x_i > x_i, x - \sum_{j=1}^{N} < x, x_j > x_j > \tag{1}$$

$$=< x, x > - \sum_{i=1}^{N} < x, x_i >< x_i, x > - \sum_{j=1}^{N} \overline{< x, x_j >} < x_j, x > \tag{2}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} < x, x_j > \overline{< x, x_j >} < x_i, x_j > \tag{3}$$

so

$$0 \leq ||x||^2 - \sum_{i=1}^{N} | < x, x_i > |^2$$

This last results holds for all N so

$$\sum_{i=1}^{\infty} | < x, x_i > |^2 \leq ||x||^2$$

Problem 4. Suppose X and Y are correlated Gaussian random variables each with mean zero and unity variance with correlation coefficient ρ.

a. Write down the joint density of (X, Y).

b. Write down the 2×2 correlation matrix $\mathbf{R_{XY}}$.

Solution:

a. The joint density of (X, Y) is

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big(-\frac{1}{2(1-\rho^2)}\big(x^2 + y^2\big) - 2\rho xy\Big)$$

b. The correlation matrix is

$$\mathbf{R_{XY}} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Problem 5. This is a continuation of Problem 4. Suppose we do not know the underlying distribution for (X, Y) but have 2-dim samples $(X_k, Y_k),\ k = 1, 2, \ldots n$ from the distribution. We wish to use these samples to estimate the correlation matrix. For this problem you may assume that $\rho = 0.5$.

To generate X and Y in Matlab we may use the following technique. First, use a Matlab function call to get a standard normal random deviate. Call this X. Now get another standard normal random deviate. Call this Z. Now let

$$Y = \rho X + \sqrt{1 - \rho^2} Z.$$

a. Verify (analytically) that Y is standard normal and $E[XY] = \rho$.

b. Using Matlab estimate the correlation matrix using $n = 50$. Then subtract the actual correlation matrix found in Problem 4 (with $\rho = 0.5$) from your estimated correlation matrix. Call this matrix result $\mathbf{E_R}$.

c. Using $n = 50$ samples in part (b) your estimate of the correlation matrix was a single result. Now repeat this process $N = 1000$ times and compute the mean and variance of each entry of your 1000 $\mathbf{E_R}$ matrices.

d. Repeat part (c) but now use $n = 500$ (N is still 1000).

e. Comment on how your mean and variance results compare when using $n = 50$ and $n = 500$.

Solution:

a. Because Y is a linear combination of two independent Gaussian random variable. Y is also a Gaussian random variable. $E[Y] = \rho E[X] + \sqrt{1 - \rho^2} E[Z] = 0$ and $\mathrm{Var}(Y) = \rho^2 \mathrm{Var}(X) + (1 - \rho^2)\mathrm{Var}(Z) = 1$. So Y is a standard norm random variable.

b.c. The solution is:

```
sample = 1000;
n = 50;
result = zeros(2,2,sample);
for i=1:sample
    X = random('norm',0,1,[n,1]);
    Z = random('norm',0,1,[n,1]);
    Y = 0.5*X+sqrt(1-0.5^2)*Z;
    R = corrcoef([X,Y]);
    R_t = [1,0.5;0.5,1];
    E_r = R-R_t;
    result(:,:,i) = E_r;
end
mean_matrix = mean(result,3)
```

```
mean_matrix =

         0   -0.0046
   -0.0046         0
```

```
var_matrix = var(result,0,3)
```

```
var_matrix =

         0    0.0114
    0.0114         0
```

```
sample = 1000;
n = 500;
result = zeros(2,2,sample);
for i=1:sample
    X = random('norm',0,1,[n,1]);
    Z = random('norm',0,1,[n,1]);
    Y = 0.5*X+sqrt(1-0.5^2)*Z;
    R = corrcoef([X,Y]);
    R_t = [1,0.5;0.5,1];
    E_r = R-R_t;
    result(:,:,i) = E_r;
end
mean_matrix = mean(result,3)
```

```
mean_matrix =

   1.0e-03 *
         0   -0.4252
   -0.4252         0
```

```
var_matrix = var(result,0,3)
```

```
var_matrix =

         0    0.0011
    0.0011         0
```

d. The mean and variance both get smaller when n is larger. This is basically the central limit theorem.