EE 562

Homework 4

Due Wednesday, February 15, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Suppose the random sequence Z_n converges to Z in distribution where Z is some constant z_0 and each Z_n is a continuous random variable. In this case show that $Z_n \to Z$ in distribution implies $Z_n \to Z$ in probability.

Solution:
Let $\epsilon > 0$

$$\begin{aligned}
P(|Z_n - Z_0| \geq \epsilon) &= P[(Z_n - Z_0) \geq \epsilon] + P[-(Z_n - Z_0) \leq \epsilon] \\
&= 1 - P(Z_n < Z_0 + \epsilon) + P(Z_n \leq Z_0 - \epsilon) \\
&= 1 - P(Z_n \leq Z_0 + \epsilon) + P(Z_n = Z_0 + \epsilon) + P(Z_n \leq Z_0 - \epsilon) \\
&= 1 - F_{Z_n}(Z_0 + \epsilon) + F_{Z_n}(Z_0 - \epsilon) \\
&\xrightarrow{n \to \infty} \\
&= 1 - 1 + 0 = 0
\end{aligned}$$

So $Z_n \to Z$ in probability. Note from the proof we see the result holds even if Z_n is not continuous.

Problem 2. Let $X_n,\ n \geq 1$, denote a sequence of independent and identically distributed zero-mean unit-variance Gaussian random variables.

Define for $n \geq 1$

$$Y_n = (1 - \alpha)X_n + \alpha Y_{n-1}$$

where $Y_0 = 0$ and $0 < \alpha < 1$. Since each Y_n is a linear combination of Gaussian random variables we know each Y_n is a Gaussian random variable. Find the limiting mean and variance of this Gaussian random sequence, i.e., compute the mean and variance of Y_n as $n \to \infty$.

Solution:

$$Y_0 = 0$$
$$Y_1 = (1-\alpha)X_1 + \alpha Y_0 = (1-\alpha)X_1$$
$$Y_2 = (1-\alpha)X_2 + \alpha Y_1 = (1-\alpha)X_2 + \alpha(1-\alpha)X_1$$
$$Y_3 = (1-\alpha)X_3 + \alpha Y_2 = (1-\alpha)X_3 + \alpha(1-\alpha)X_2 + \alpha^2(1-\alpha)X_1$$

continuing we find

$$Y_n = (1-\alpha)\sum_{k=0}^{n} \alpha^k X_{n-k}$$

We find

$$E[Y_n] = (1-\alpha)\sum_{k=0}^{n} \alpha^k E[X_{n-k}] = 0$$

and

$$Var[Y_n] = (1-\alpha)^2 \sum_{k=0}^{n} \alpha^{2k} Var[X_{n-k}] = (1-\alpha)^2 \sum_{k=0}^{n} \alpha^{2k} \cdot 1 = \frac{1-\alpha}{1+\alpha}$$

Problem 3. Design a correlation detector for deciding whether H_0 or H_1 is true where

$$H_i: \ \mathbf{X}(u) = \mathbf{S}_i + \mathbf{N}(u), \quad i = 0, 1$$

and

$$\mathbf{S}_i = (-1)^i \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$E[\mathbf{N}(u)] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{K_N} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}.$$

Indicate the decision regions on a graph.

Solution:

We first redefine the hypotheses so that the noise has zero mean:

$$H_0' : \mathbf{X}(u) = \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) + \mathbf{N}'(u) = \mathbf{S}_0' + \mathbf{N}'(u)$$
$$H_1' : \mathbf{X}(u) = \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) + \mathbf{N}'(u) = \mathbf{S}_1' + \mathbf{N}'(u)$$

where $\mathbf{N}'(u) \sim \mathcal{N}_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4 & -1\\-1 & 4\end{bmatrix}\right)$. The threshold is calculated as

$$T = \frac{\mathbf{S_1'^\dagger K_N^{-1} S_1' - S_0'^\dagger K_N^{-1} S_0'}}{2} = -\frac{4}{3}$$

In order to compute the decision region we calculate

$$x^\dagger(u)\mathbf{K}_n^{-1}(\mathbf{S}_1' - \mathbf{S}_0') \overset{H_0}{<} -\frac{4}{3}$$

$$\Rightarrow \begin{bmatrix}x_1 & x_2\end{bmatrix} \mathbf{K_N}^{-1} \begin{bmatrix}-2\\0\end{bmatrix} \overset{H_0}{<} -\frac{4}{3}$$

$$\Rightarrow 4x_1 + x_2 \overset{H_0}{>} 10$$

The decision boundary is a straight line that intercepts the x_1-axis at 2.5 and intercepts the x_2-axis at 10.

Problem 4. We are given an observation of X and we must decide between two hypotheses:

$$H_0: \quad X = N$$

$$H_1: \quad X = S_1 + N$$

where, N is a continuous random variable with density

$$f_N(n) = \begin{cases} 2e^{-2n}, & n > 0, \\ 0, & \text{elsewhere} \end{cases}$$

and $S_1 = 2.1$.

a. Find a threshold T so that the probability of type I error is 0.01.

b. Using the T you found find the probability of type II error.

c. What is the power of this test?

Solution:

a. Type I error $= P(S(X) = 1|H_0 \text{ is ture}) = \int_T^\infty f(x)\,\mathrm{d}x = \int_T^\infty 2e^{-2n}\,\mathrm{d}n = 0.01 \Rightarrow T \approx 2.3$

b. Type II error $=P(S(X)=0|H_1 \text{ is ture}) = \int_0^{0.2} 2e^{-2n}\,\mathrm{d}n = 0.33$

c. Power of test = 1-Type II error = 1-0.33=0.67

Problem 5. We are given an observation of X and we must decide between two hypotheses:

$$H_0: \quad X = N$$
$$H_1: \quad X = S_1 + N$$

where, N is a continuous random variable with density

$$f_N(n) = \begin{cases} n+1, & -1 \leq n \leq 0, \\ 1-n, & 0 < n \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find a threshold T (to 3 decimal places) so that the probability of type I error is 0.001.

b. Find the value of S_1 so that the probability of type II error is 0.01.

Solution:

a. $P(\text{Type I error}) = \int_T^1 (1-n)\,\mathrm{d}n = \frac{1}{2}(T-1)^2 = 0.001 \Rightarrow T \approx 0.955$

b. $P(\text{Type II error}) = \int_{-1}^{T-S_1} (n+1)\,\mathrm{d}n = \frac{1}{2}(1+T-S_1)^2 = 0.01 \Rightarrow S_1 \approx$ 1.81