EE 562

Homework 3

Due Wednesday, February 8, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Suppose the continuous random variable U is uniform on [0,1]. Determine for each of the following if the sequence converges and, if so, in what sense and to what limiting random variable.

a. $X_n(u) = u^n$.

b. $Z_n(u) = cos^n(2\pi u)$.

Solution:

a. For all $u \neq 1$, $\lim_{n\to\infty} X_n(u) = 0$. $\lim_{n\to\infty} X_n(u) = 1$ only when $u = 1$. The probability of $0 \leq u < 1$ is 1. So $X_n(u) \to 0$ almost surely.

b. $\lim_{n\to\infty} Z_n(u) \to 0$ except for $u = 0, \frac{1}{2}, 1$. Let $A = (0, 0.5) \bigcup (0.5, 1)$. We know $Pr(U \in A) = 1$. So $Z_n(u) \to 0$ almost surely.

Problem 2. Let X_n be a sequence of i.i.d. equiprobable Bernoulli (0,1) random variables and let

$$Y_n = 2^n X_1 X_2 \cdots X_n.$$

a. Does Y_n converge almost surely and, if so, to what limit?

b. Does Y_n converge in the mean square sense and, if so, to what limit?

Solution:

a. Let denote a new random variable as $A_n = X_1 X_2 \ldots X_n$. A_n has only two values, 0 or 1. We know that $Pr(A_n = 1) = \frac{1}{2^n}$, which means, $\lim_{n\to\infty} Pr(A_n = 1) = 0$. And $\lim_{n\to\infty} Pr(A_n = 0) = 1$. So $Y_n \to 0$ almost surely.

b. From (a) we know

$$Pr(A_n) = \begin{cases} 1 - \frac{1}{2^n} & A_n = 0 \\ \frac{1}{2^n} & A_n = 1 \end{cases}$$

Then we have

$$E\left[Y_n^2\right] = \frac{1}{2^n} \times 2^{2n} = 2^n$$

And assume $m > n$, there is

$$E\left[Y_n Y_m\right] = (2^n)(2^m)/2^m = 2^n$$

To verify whether Y_n converge in the mean square sense, we need to calculate

$$\begin{aligned} \lim_{n,m\to\infty} E\left[|Y_n - Y_m|^2\right] &= \lim_{n,m\to\infty} E\left[Y_n^2\right] + E\left[Y_m^2\right] - 2E\left[Y_n Y_m\right] \\ &= \lim_{n,m\to\infty} 2^n = \infty \end{aligned}$$

which means Y_n does not converge in mean square sense.

Problem 3. Let X_n be a sequence of i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$. Let

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

Show that Y_k converges in the mean square sense.

Solution:

First, we need to calculate

$$\begin{aligned} E[Y_n] &= \frac{1}{n} \sum_{k=1}^{n} X_k = \mu \\ E[Y_n^2] &= \frac{1}{n^2} \left[\sum_{k=1}^{n} E\left(X_k^2\right) + n^2\mu^2 \right] \end{aligned}$$

Then we can calculate

$$\begin{aligned} E\left[|Y_n - \mu|^2\right] &= \frac{1}{n^2}\sum_{k=1}^{n} E\left[X_k^2\right] + \mu^2 - 2\mu E\left[Y_n\right] + \mu^2 \\ &= \frac{\sum_{k=1}^{n} E\left[X_k^2\right]}{n^2} \\ &= \frac{\sigma^2 + \mu^2}{n^2} \end{aligned}$$

Because $\sigma^2 < \infty$, we know

$$\lim_{n\to\infty} E\left[|Y_n - \mu|^2\right] = 0$$

So Y_n converges to μ in the mean square sense.

Problem 4. Let X_n and Y_n denote two (possibly dependent) sequences of random variables that converge in the mean square sense to X and Y, respectively. Does the sequence $Z_n = X_n + Y_n$ converge in the mean square sense and, if so, to what limit?

Solution:
We can start by look at

$$\begin{aligned} \lim_{n\to\infty} E\left[|Z_n - X - Y|^2\right] = \lim_{n\to\infty} E\left[(X_n - X)^2\right] + E\left[(Y_n - Y)^2\right] \\ + 2E\left[(X_n - X)(Y_n - Y)\right] \end{aligned}$$

Because X_n and Y_n converge to X and Y in mean square sense. The first two terms in the right hand side of above equation equal 0. By Cauchy–Schwarz inequality, we also have

$$\left|E\left[(X_n - X)(Y_n - Y)\right]\right| \le \sqrt{E\left[(X_n - X)^2\right] E\left[(Y_n - Y)^2\right]}$$

which implies

$$\lim_{n\to\infty} E\left[(X_n - X)(Y_n - Y)\right] = 0$$

So Z_n converges to $X + Y$ in mean square sense.

Problem 5. Suppose U is uniform on [0,1]. Define

$$X_n(u) = \sum_{k=1}^{n} \frac{1}{u+k}.$$

Does $X_n(u)$ converge and, if so, in what sense and to what limit?
Solution:
We know, for all $u \in [0,1]$, $X_n(u) \geq \sum_{k=1}^{n} \frac{1}{k+1}$. This implies

$$\lim_{n\to\infty} X_n(u) \geq \lim_{n\to\infty} \sum_{k=2}^{n+1} \frac{1}{k} = \infty$$

This means, with probability one, $X_n(u)$ diverges when $n \to \infty$. We also know that, for $n \to \infty$, $X_n(u) > 1$, which means

$$\lim_{n\to\infty} X_n^2(u) > \lim_{n\to\infty} X_n(u) = \infty$$

Then, for any finite random variable Z, we have

$$\lim_{n\to\infty} E\left[|X_n(u) - Z|^2\right] \geq \lim_{n\to\infty} E\left[|X_n(u)|^2 - |Z|^2\right] = \infty$$

So $X_n(u)$ also diverges in mean square sense. $X_n(u)$ does not converge.