EE 562

Homework 2

Due Wednesday, February 1, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Are there any real values of β that would permit the following matrix to be a correlation matrix? If so, find them. If not, show why not.

$$A = \begin{bmatrix} 3-\beta & 2 \\ 2 & 1+\beta \end{bmatrix}.$$

Solution:
There is only one such β. To see this, we need to do an eigen-decomposition on matrix $\mathbf{A}$. because $\mathbf{A}$ is positive semi-definite, all the two eigenvalues should be non-negative. The eigen-equation is

$$\mathbf{A} - \lambda \mathbf{I} = 0$$

which can be written explicitly

$$\begin{vmatrix} 3-\beta-\lambda & 2 \\ 2 & 1+\beta-\lambda \end{vmatrix} = 0$$

By solving the equation, we know the two eigenvalues are

$$\lambda_1 = 2 - \sqrt{5 - 2\beta + \beta^2} \quad \lambda_2 = 2 + \sqrt{5 - 2\beta + \beta^2}$$

Because $\lambda_1 \geq 0, \lambda_2 \geq 0$, we have

$$2 \geq \sqrt{5 - 2\beta + \beta^2} \Longrightarrow (\beta - 1)^2 \leq 0$$

The only β which satisfy the inequality is $\beta = 1$.

Problem 2. Let $\mathbf{W}(u)$ be a white random vector with

$$\mu_W = (0 \quad 0 \quad 0)^t, \quad \mathbf{K}_W = \mathbf{I}.$$

Let

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{c}.$$

Find **c** and a causal matrix **H** using the direct method that produces

$$\mu_X = [1 \quad 2 \quad 3]^T, \quad \mathbf{K}_X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 6 \end{bmatrix}.$$

Solution: First we can calculate $\mu_{\mathbf{X}}$

$$\mu_{\mathbf{X}} = E(\mathbf{HW} + c) \Longrightarrow \mu_{\mathbf{X}} = \mathbf{H}\mu_{\mathbf{W}} + c$$

So, $c = \mu_{\mathbf{X}} = [0 \quad 0 \quad 0]^T$.
From the previous homework, we know $\mathbf{K_X} = \mathbf{HK_W H}^\dagger = \mathbf{HH}^\dagger$. The procedure of direct method is

$$\begin{aligned}
k_{11} &= |h_{11}|^2 \Longrightarrow h_{11} = 1 \\
k_{12} &= h_{11}h_{21}^* \Longrightarrow h_{21} = 1 \\
k_{13} &= h_{11}h_{31}^* \Longrightarrow h_{31} = 1 \\
k_{22} &= |h_{21}|^2 + |h_{22}|^2 \Longrightarrow h_{22} = 1 \\
k_{23} &= h_{21}h_{31}^* + h_{22}h_{32}^* \Longrightarrow h_{32} = 1 \\
k_{33} &= |h_{11}|^2 + |h_{22}|^2 + |h_{33}|^2 \Longrightarrow h_{33} = 2
\end{aligned}$$

So the matrix **H** is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Problem 3. Stark and Woods 5.29.
Solution:
Mean:

$$\begin{aligned}
E[Y] &= \mathbf{A^T} E[\mathbf{X}] + B \\
&= 32
\end{aligned}$$

Variance:

$$\begin{aligned}
\sigma_Y &= \mathbf{A}^T \mathbf{K_X} \mathbf{A} \\
&= 25
\end{aligned}$$

Problem 4. Stark and Woods 5.34.
Solution: Let $x(\mu)$ be an n-dimensional vector of mean-zero real Gaussian random variables. The expected value of the product of the random variables in this vector can be computed by appropriate differentiation of the characteristic function of this Gaussian random vector.

$$E\left[\prod_{t=1}^{n} x(\mu,t)\right] = (-i)^n \frac{\partial^n}{\partial v_1 \ldots \partial v_n} E\left[\exp\left(i\sum_{t=1}^{n} v_t x(\mu,t)\right)\right]\Bigg|_{\mathbf{v}=0}$$

The characteristic function can be substituted into this expression and the partial derivative calculated in an organized fashion.

$$\begin{aligned} E\left[\prod_{t=1}^{n} x(\mu,t)\right] &= (-i)^n \left[\frac{\partial^n}{\partial v_1 \ldots \partial v_n} \exp\bigl(-\mathbf{v}^{\mathbf{t}}\mathbf{K_x}\mathbf{v}/2\bigr)\right]\Bigg|_{\mathbf{v}=0} \\ &= (-i)^n \left[\frac{\partial^n}{\partial v_1 \ldots \partial v_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \bigl(-\mathbf{v}^{\mathbf{t}}\mathbf{K_x}\mathbf{v}\bigr)^m\right]\Bigg|_{\mathbf{v}=0} \\ &= (-i)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \Bigg[\frac{\partial^n}{\partial v_1 \ldots \partial v_n} \left(\sum_{j_1=1}^{n}\sum_{k_1=1}^{n} v_{j_1} v_{k_1} \mathbf{K_x}(j_1,k_1)\right) \\ &\qquad \ldots \left(\sum_{j_m=1}^{n}\sum_{k_m=1}^{n} v_{j_m} v_{k_m} \mathbf{K_x}(j_m,k_m)\right)\Bigg]\Bigg|_{\mathbf{v}=0} \end{aligned}$$

Now the differentiation process reduces to differentiating products of the form

$$v_{j_1} v_{k_1} v_{j_2} v_{k_2} \ldots v_{j_m} v_{k_m} = \prod_{t=1}^{n} v_t^{n_t}$$

where n_t simply counts the number o times that v_t occurs in the product on the left side. It follows immediately that

$$\left[\frac{\partial^n}{\partial v_1 \ldots \partial v_n} v_{j_1} v_{k_1} v_{j_2} v_{k_2} \ldots v_{j_m} v_{k_m}\right]\Bigg|_{\mathbf{v}=0} = \prod_{t=1}^{n} \left[\frac{\partial}{\partial v_t} v_t^{n_t}\right]\Bigg|_{\mathbf{v}=0}$$

Only the term with $m = n/2$ can be non-zero because each of the m quadratic forms in second equation contributes two v variables, and furthermore, the expected value of the product must be zero when n is odd. Continuing on the case in which n is even, the second equation can be reduced using the third one.

$$E\left[\prod_{t=1}^{n} x(\mu,t)\right] = \frac{1}{2^{n/2}(n/2)!} \sum_{\left(j_1,k_1,\ldots,j_{n/2},k_{n/2}\right)\in\mathcal{I}} \prod_{s=1}^{n/2} \mathbf{K_x}(j_s,k_s)$$

where $\mathcal{I}$ is the set of n-tuple in which each integer from 1 to n occurs exactly once. The above expression can be simplified further because the same factors occur in the product for different n-tuples from $\mathcal{I}$. In fact, each possible product occurs exactly $2^{n/2}(n/2)!$ times one the right hand side of the expression. Taking advantage of this property leads to the following end results:
Let $x(\mu,t), t = 1,2,\ldots,n$ be mean-zero jointly Gaussian, real random variables. Let $\mathcal{J}$ be the collection of all unordered sequences of all unordered pairs, wich each sequence containing each integer from 1 to n exactly once. Than, for even n,

$$E\left[\prod_{t=1}^{n} x(\mu,t)\right] = \sum_{\left(j_1,k_1,\ldots,j_{n/2},k_{n/2}\right)\in\mathcal{J}} \prod_{s=1}^{n/2} \mathbf{K_x}(j_s,k_s)$$

The number of n possible pairings of n integers, i.e., the size of $\mathcal{J}$, is $\frac{n!}{2^{n/2}(n/2)!}$, and this number grows quite rapidly as n increases through the even integers. For example, for real mean-zero Gaussian random variables

$$E\left[\prod_{t=1}^{4} x(\mu,t)\right] = \mathbf{K_x}(1,2)\mathbf{K_x}(3,4) + \mathbf{K_x}(1,3)\mathbf{K_x}(2,4) + \mathbf{K_x}(1,4)\mathbf{K_x}(2,3)$$

Problem 5. Let $\mathbf{X}(u)$ be an n-dimensional random vector with covariance matrix $\mathbf{K}_X$ and correlation matrix $\mathbf{R}_X$. Let $(\lambda_i,\ \mathbf{e}_i),\ i = 1,2,\ldots,n$ denote the eigenvalue and eigenvector pairs of the covariance matrix $\mathbf{K}_X$, with the eigenvectors chosen to form an orthonormal set.

a. If $\mathbf{K}_X$ is non-singular, can $\mathbf{R}_X$ be singular? Why or why not?

b. Show $\mathbf{K}_X$ can be written in the form

$$\mathbf{K}_X = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \mathbf{e}_i^{\dagger}.$$

c. Construct an example in which $\mathbf{R}_X$ is non-singular but $\mathbf{K}_X$ is singular.

d. Construct an example in which $\mathbf{K}_X$ and $\mathbf{R}_X$ are both singular but $\mu_X \neq 0$.

e. Verify that

$$\mathbf{R}_X^{-1} = \mathbf{K}_X^{-1} - \frac{\mathbf{K}_X^{-1}\mu_X\mu_X^\dagger\mathbf{K}_X^{-1}}{1 + \mu_X^\dagger\mathbf{K}_X^{-1}\mu_X}.$$

Solution:

a.

Let $\mathbf{K_X} = \mathbf{E\Lambda E}^\dagger$ and let $\mu_\mathbf{X} = \sum_{i=1}^n \mu_i\mathbf{e}_i$. Since $\{\mathbf{e}_i\}_{i=1}^n$ span the space of n dimensional column vectors this representation of $\mu_\mathbf{X}$ is always possible. Using this fact,

$$\begin{aligned}\mathbf{R_X} &= \mathbf{E\Lambda E}^\dagger + \mu_\mathbf{X}\mu_\mathbf{X}^\dagger \\ &= \mathbf{E\Lambda' E}^\dagger\end{aligned}$$

where $\mathbf{\Lambda}'$ is the diagonal matrix with $\lambda_i' = \lambda_i + \|\mu_i\|^2$. If $\lambda_i \neq 0$ then $\lambda_i' \neq 0$ since $\|\mu_i\|^2 > 0$. Thus, if $\mathbf{K_X}$ is nonsingular, $\mathbf{R_X}$ is also nonsingular.

b.

Again let $\mathbf{K_X} = \mathbf{E\Lambda E}\dagger$. Let $\mathbf{\Lambda}_i$ be the matrix with the (i,i)-th element equal to λ_i and all other elements equal to zero. Clearly, $\mathbf{\Lambda} = \sum_{i=1}^n \mathbf{\Lambda}_i$. Using this fact,

$$\begin{aligned}\mathbf{K_X} &= \mathbf{E\Lambda E}\dagger \\ &= \mathbf{E}\left(\sum_{i=1}^n \mathbf{\Lambda}_i\right)\mathbf{E}\dagger \\ &= \mathbf{E}\left(\sum_{i=1}^n \lambda_i\mathbf{e}_i^\dagger\right) \\ &= \sum_{i=1}^n \lambda_i\mathbf{e}_i\mathbf{e}_i^\dagger\end{aligned}$$

c,d.

We can use what we learned in part a) to find a simple example. Consider the following matrix:

$$\mathbf{K_X} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

which has the eigen-decomposition $\lambda_1 = 10, \mathbf{e}_1 = \frac{1}{\sqrt{10}}[3,1]^T; \lambda_2 = 0, \mathbf{e}_2 = \frac{1}{\sqrt{10}}[1,-3]^T$. Since $\lambda_2 = 0$,$\mathbf{K_X}$ is singular. If we let $\mu_\mathbf{X} = \sqrt{10}\mathbf{e}_2$ then:

$$\mathbf{R_X} = \mathbf{K_X} + \mu_\mathbf{X}\mu_\mathbf{X}^\dagger = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

which is non-singular. Alternatively, if we let $\mu_\mathbf{X} = \sqrt{10}\mathbf{e}_1$ then:

$$\mathbf{R_X} = \mathbf{K_X} + \mu_\mathbf{X}\mu_\mathbf{X}^\dagger = \begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix}$$

which is singular.

e.

We want to show that

$$\mathbf{R_X^{-1}} = \mathbf{K_X^{-1}} - \frac{\mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}}{\mathbf{1} + \mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}}$$

We first multiply both sides by $\mathbf{R_X} = \mathbf{K_X} + \mu_\mathbf{X}\mu_\mathbf{X}^\dagger$ to get:

$$\mathbf{I} = \mathbf{I} + \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger - \frac{\mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger + \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger}{\mathbf{1} + \mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}}$$

Since $\mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}$ is a scalar, we can factor to obtain:

$$\begin{aligned}\mathbf{I} &= \mathbf{I} + \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger - \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger\frac{\mathbf{1} + \mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}}{\mathbf{1} + \mu_\mathbf{X}^\dagger\mathbf{K_X^{-1}}\mu_\mathbf{X}} \\ &= \mathbf{I} + \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger - \mathbf{K_X^{-1}}\mu_\mathbf{X}\mu_\mathbf{X}^\dagger \\ &= \mathbf{I}\end{aligned}$$

which completes the proof.