EE 562

Homework 1

Due Wednesday, January 25, 2017 at 6:40 p.m.

Work all 5 problems.

Problem 1. Suppose the random variable X has density

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Compute $P(X < 1)$.

b. Let $Y = X^2 + 1$. Find $E(Y)$.

Solution:

a.

$$\begin{aligned} P(X < 1) &= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \\ &= 2\int_0^1 e^{-2x}\,\mathrm{d}x \\ &= \frac{2}{-2}e^{-2x}\Big|_0^1 \\ &= 1 - e^{-2} \end{aligned}$$

b.

$$\begin{aligned} E(Y) &= E\big(X^2 + 1\big) \\ &= E\big(X^2\big) + 1 \end{aligned}$$

The problem becomes finding $E(X^2)$.

$$\begin{aligned} E\big(X^2\big) &= 2\int_0^{\infty} x^2 e^{-2x}\,\mathrm{d}x \\ &= -\left(x^2 e^{-2x}\Big|_0^{\infty} - 2\int_0^{\infty} x e^{-2x}\,\mathrm{d}x\right) \\ &= \frac{1}{2} \end{aligned}$$

So, $E(Y) = 1 + \frac{1}{2} = \frac{3}{2}$.

Problem 2.

a. Suppose X is exponentially distributed with parameter λ, i.e.,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Derive the moment generating function for X.

b. Suppose the random variable Y has moment generating function

$$M_Y(s) = (1-2s)^{-1}, \quad s < \frac{1}{2}.$$

Find the mean and variance of Y.

Solution:

a.

Moment generating function for X is

$$\begin{aligned} g_X(r) &= E\left[e^{rX}\right] \\ &= \int_0^\infty \lambda e^{rx} e^{-\lambda x}\,\mathrm{d}x \\ &= \frac{\lambda}{r-\lambda} e^{(r-\lambda)x}\Big|_0^\infty \\ &= \frac{\lambda}{\lambda - r} \quad (r < \lambda) \end{aligned}$$

b.

From the definition of MGF, we know

$$\frac{\mathrm{d}^k g_X(r)}{\mathrm{d}r^k}\bigg|_{r=0} = E\left[X^k\right]$$

So we have

$$\begin{aligned} E[Y] &= \frac{2}{(1-2s)^2}\bigg|_{s=0} \\ &= 2 \end{aligned}$$

$$
\begin{aligned}
E\left[Y^2\right] &= \left.\frac{8}{(1-2s)^3}\right|_{s=0} \\
&= 8
\end{aligned}
$$

The variance of Y is

$$\operatorname{Var}(Y) = E\left[Y^2\right] - E[Y]^2 = 4$$

Problem 3. Compute the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Solution:
The eigenequation for the matrix is

$$\det\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

which give

$$(\lambda-3)(\lambda-4)(\lambda-1) = 0$$

So the eigenvalues of the matrix are $\lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 1$. Then, the eigenvectors can be solved by equations:

$$\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i \quad (i = 1, 2, 3)$$

The eigenvectors are $\mathbf{e}_1 = [-1, 0, 1]^T$, $\mathbf{e}_2 = [1, -1, 1]^T$ and $\mathbf{e}_3 = [1, 2, 1]^T$.

Problem 4. Suppose $Y(u) = A^T X(u) + b$ where

$$A^T = [2 \quad -1 \quad 2], \quad \mu_X^T = [5 \quad -5 \quad 6], \quad b = 5$$

and let

$$K_X = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

a. Compute the mean of $Y(u)$.

b. Compute the variance of $Y(u)$.

Solution:
a.

$$\begin{aligned}
E[Y(u)] &= E\big[A^T\mathbf{X}(u) + b\big] \\
&= A^T E[\mathbf{X}(u)] + b \\
&= A^T \mu_{\mathbf{X}} + b \\
&= \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix} + 5 \\
&= 32
\end{aligned}$$

b.

$$\begin{aligned}
\mathrm{Var}(Y) &= E\big[\big(A^T\mathbf{X} + b - \mu_Y\big)\big(A^T\mathbf{X} + b - \mu_Y\big)\big] \\
&= E\Big[A^T(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^T A\Big] \\
&= A^T \mathbf{K}_{\mathbf{X}} A \\
&= 25
\end{aligned}$$

Problem 5. Suppose $\mathbf{Y}(u)$ is formed by a linear transformation of $\mathbf{X}(u)$. Here $\mathbf{X}(u) \in \mathbf{R}^n$ and $\mathbf{Y}(u) \in \mathbf{R}^m$.

$$\mathbf{Y}(u) = \mathbf{H}\mathbf{X}(u)$$

where

$$H = \begin{bmatrix} h_{11} & \dots & h_{1n} \\ \vdots & \dots & \vdots \\ h_{m1} & \dots & h_{mn} \end{bmatrix}.$$

Show $\mathbf{K}_{\mathbf{Y}} = \mathbf{H}\mathbf{K}_{\mathbf{X}}\mathbf{H}^{\dagger}$.

Solution:

$$
\begin{aligned}
\mathbf{K_Y} &= E\Big[(\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Y} - \mu_{\mathbf{Y}})^{\dagger}\Big] \\
&= E\Big[(\mathbf{HX} - \mu_{\mathbf{X}})(\mathbf{HX} - \mu_{\mathbf{X}})^{\dagger}\Big] \\
&= E\Big[\mathbf{H}(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\dagger}\mathbf{H}^{\dagger}\Big] \\
&= \mathbf{H}E\Big[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\dagger}\Big]\mathbf{H}\dagger \\
&= \mathbf{HK_X H}^{\dagger}
\end{aligned}
$$