26.0 Miscellaneous Topics

26.1 The Poisson Process

Consider a sequence of i.i.d. random variables $\gamma(n),\ n \geq 1$, with density

$$f_\gamma(t,n) = \lambda e^{-\lambda t} u(t), \quad n = 1, 2, \ldots .$$

Define

$$T(n) = \sum_{k=1}^{n} \gamma(k).$$

$T(n)$ would represent the time of arrival of the nth event if $\gamma(n)$ represents the interarrival times. This is used in modeling counts on a Geiger counter that detects particles and is also used in Queuing theory.

Now $T(n)$ is the sum of n i.i.d. random variables so its pdf is the $(n-1)$ fold convolution of $f_\gamma(t,n)$. We get

$$f_T(t,n) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} u(t)$$

$$E[T(n)] = E\left[\sum_{k=1}^{n} \gamma(k)\right] = n/\lambda$$

$$Var[T(n)] = n/\lambda^2 = nVar[\gamma(n)].$$

Define

$$N(t) = \sum_{n=1}^{\infty} u[t - T(n)]$$

which equals the number of arrivals (or events) up to and including time t. Now

$$\gamma(n) = T(n) - T(n-1)$$

$$P[N(t) = n] = P[T(n) \leq t,\ T(n+1) > t]$$

$$= P[T(n) \leq t,\ \gamma(n+1) > t - T(n)]$$

$$= \int_0^t f_T(\alpha, n) \int_{t-\alpha}^{\infty} f_\gamma(\beta, n+1) d\beta d\alpha$$

$$= \int_0^t \frac{\lambda^n \alpha^{n-1} e^{-\lambda t}}{(n-1)!} \int_{t-\alpha}^{\infty} \lambda e^{-\lambda \beta} d\beta d\alpha \ u(t)$$

$$= \left(\int_0^t \alpha^{n-1} d\alpha \right) \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \, u(t)$$

or

$$P[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \, u(t), \quad t \geq 0.$$

Note that $T(n)$ has independent increments. So

$$P[N(t_b) - N(t_a) = n] = \frac{[\lambda(t_b - t_a)]^n}{n!} e^{-\lambda(t_b - t_a)} \, u(n).$$

Now

$$E[N(t)] = \lambda t.$$

Suppose $t_2 \geq t_1$. Then

$$E[N(t_2)N(t_1)] = E[(N(t_1) + [N(t_2) - N(t_1)])N(t_1)]$$

$$= E[(N(t_1)^2] + E[N(t_2) - N(t_1)]E[N(t_1)]$$

$$\lambda t_1 + \lambda^2 t_1^2 + \lambda(t_2 - t_1)\lambda t_1$$

$$= \lambda t_1 + \lambda^2 t_1 t_2.$$

For $t_1 > t_2$ a similar expression holds. Thus,

$$R_N(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$

and

$$K_N(t_1, t_2) = \lambda \min(t_1, t_2).$$

Example: Radioactivity monitoring that counts particles can often be modeled as Poisson. We start monitoring at time t and count for T_0 seconds.

Let ΔN = number of counts in the interval $[t, t + \tau] = N(t + \tau) - N(t)$. Then ΔN has a Poisson distribution with mean λT_0 where λ is the average arrival rate. The probability that an alarm does not sound is

$$P[\Delta N \leq N_0] = \sum_{k=0}^{N_0} \frac{(\lambda T_0)^k}{k!} e^{-\lambda T_0}.$$

Sampling Theorem for Bandlimited WSS Random Processes

Definition: A WSS random process $X(u,T)$ is said to be *bandlimited* to $[\omega_1, \omega_2]$ if $S_X(\omega) = 0$ for $\omega \notin [\omega_1, \omega_2]$.

If $\omega_1 = 0$, we have a lowpass system with cutoff frequency $\omega_c = \omega_2$. Here

$$R_X(\tau) = \sum_{n=-\infty}^{\infty} R_X(nT) \frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}$$

where $T = \pi/\omega_c$.

Theorem: If a 2nd order WSS RP $X(u,t)$ is lowpass with cutoff frequency ω_c, then

$$X(u,t) = \sum_{n=-\infty}^{\infty} X(u,nT) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)} \quad (M.S.)$$

i.e., with

$$X_N(u,t) = \sum_{n=-N}^{N} X(u,nT) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}$$

then

$$\lim_{N\to\infty} E[|X(u,t) - X_N(u,t)|^2] = 0.$$

Proof:

$$E[|X(u,t) - X_N(u,t)|^2]$$
$$= E[(X(u,t) - X_N(u,t))X^*(u,t)] - E[(X(u,T) - X_N(u,t))X_N^*(u,t)].$$

Now

$$E[(X(u,t) - X_N(u,t))X^*(u,t)]$$
$$= R_X(0) - \sum_{n=-N}^{N} R_X(nT - t) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

But

$$R_X(\tau - t) = \sum_{n=-\infty}^{\infty} R_X(nT - t) \frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}.$$

Set $\tau = t$ to get

$$R_X(0) = \sum_{n=-\infty}^{\infty} R_X(nT - t) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

Thus

$$E[(X(u,t) - X_N(u,t))X^*(u,t)] \to 0 \text{ as } N \to \infty.$$

Now

$$E[(X(u,t) - X_N(u,t))X^*(u,mT)]$$
$$= R_X(t - mT) - \sum_{n=-N}^{N} R_X(nT - mT)\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

But

$$R_X(\tau - mT) = \sum_{n=-\infty}^{\infty} R_X(nT - mT)\frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}.$$

Set $\tau = t$ to get

$$R_X(t - mT) = \sum_{n=-\infty}^{\infty} R_X(nT - mT)\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

Thus

$$E[(X(u,t) - X_N(u,t))X^*(u,mT)] \to 0 \text{ as } N \to \infty.$$

Next let

$$\hat{X}(u,t) = \lim_{N\to\infty} X_N(u,t) \quad (M.S.).$$

Then

$$E[(X(u,t) - \hat{X}(u,t))X^*(u,mT)] = 0.$$

But, $X_N(u,t)$ is a linear combination of $X(u,mT)$ for $m = -N$ to N so

$$E[(X(u,t) - \hat{X}(u,t))X_N^*(u,t)] = 0$$

which implies

$$\lim_{N\to\infty} E[(X(u,t) - X_N(u,t))X_N^*(u,t)] = 0.$$

So

$$\lim_{N\to\infty} E[|X(u,t) - X_N(u,t)|^2] = 0.$$