25.0 Bandpass Systems

25.1 Representations of Bandpass Systems

Let $s(t)$ be real-valued.

$$s(t) = a(t)\cos[2\pi f_c t + \theta(t)] \qquad (1)$$

$$a(t) = \textit{amplitude (or envelope) of} \, s(t)$$
$$\theta(t) = \textit{phase of} \, s(t)$$
$$f_c = \textit{carrier frequency of} \, s(t)$$

If bandwidth is much smaller than f_c, we have a bandpass system.

$$s(t) = a(t)\cos(\theta(t))\cos(2\pi f_c t) - a(t)\sin(\theta(t))\sin(2\pi f_c t)$$
$$= x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t) \qquad (2)$$
$$x(t) = a(t)\cos(\theta(t)) \quad \longrightarrow \quad \textit{in phase component}$$
$$y(t) = a(t)\sin(\theta(t)) \quad \longrightarrow \quad \textit{quadrature component}$$

$x(t)$ and $y(t)$ are low-pass signals, since their frequency component is concentrated around $f = 0$.
Let

$$u(t) = a(t)e^{i\theta(t)}$$
$$= x(t) + iy(t)$$

Then,

$$s(t) = Re\{u(t)e^{i2\pi f_c t}\} \qquad (3)$$

So, $s(t)$ has the 3 representations shown above in $(1), (2)$ and (3)

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi f t}dt$$
$$= \int_{-\infty}^{\infty} \{Re[u(t)e^{i2\pi f_c t}]\}e^{-i2\pi f t}dt$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} [u(t)e^{i2\pi f_c t} + u^*(t)e^{-i2\pi f_c t}]e^{-i2\pi f t}dt$$

$$= \frac{1}{2}[U(f - f_c) + U^*(-f - f_c)]$$

where, $u(t) \overset{F.T.}{\longleftrightarrow} U(f)$

Since frequency content of $s(t)$ is concentrated around f_c, we see that the frequency content of $u(t)$ is around $f = 0$. So, the complex valued waveform $u(t)$ is a low-pass signal waveform and is called the equivalent low-pass signal

The energy in $s(t)$ is

$$\xi = \int_{-\infty}^{\infty} s^2(t)dt$$

$$= \int_{-\infty}^{\infty} \{Re[u(t)e^{i2\pi f_c t}]\}^2 dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |u(t)|^2 dt + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} |u(t)|^2 \cos[4\pi f_c t + 2\theta(t)]dt}_{small\ compared\ to\ the\ 1^{st}\ integral}$$

So,

$$\xi \approx \frac{1}{2}\int_{-\infty}^{\infty} |u(t)|^2 dt$$

where, $|u(t)| = a(t)$, the envelope.

25.2 Representations of Linear Bandpass Systems

Here $h(t)$ is real, so

$$H^*(-f) = H(f)$$

Define,

$$C(f - f_c) = \begin{cases} H(f), & f > 0 \\ 0, & f < 0. \end{cases}$$

Then,

$$C^*(-f - f_c) = \begin{cases} 0, & f > 0 \\ H^*(-f), & f < 0. \end{cases}$$

So,

$$H(f) = C(f - f_c) + C^*(-f - f_c)$$

$$\Longrightarrow h(t) = c(t)e^{i2\pi f_c t} + c^*(t)e^{-i2\pi f_c t}$$
$$= 2Re[c(t)e^{i2\pi f_c t}]$$

Here, $c(t)$ is the impulse response of the equivalent low-pass system and is complex.

A filter that is encounted in the generation of single-sideband signal has the impulse resonse,

$$h(t) = \frac{1}{\pi t}$$

$$\Longrightarrow H(f) = \begin{cases} -i, & f > 0 \\ i, & f < 0. \end{cases}$$

$H(f)$ represents an all-pass filter which introduces a -90^o phase shift for $f < 0$.
The output is (for input $s(t)$)

$$r(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau$$

This is called a Hilbert transform $\longrightarrow$ output $= \hat{s}(t)$

25.3 Response of a Bandpass System to a Bandpass Signal

So far we have seen that a narrowband bandpass signal and system can be represented by equivalent low-pass signals and systems.

We want to look at the output,

$$s(t) = Re[u(t)e^{i2\pi f_c t}]$$

$$h(t) = 2Re[c(t)e^{i2\pi f_c t}]$$

$$r(t) = Re[v(t)e^{i2\pi f_c t}], \qquad some\ v(t)$$

where,

$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

$$R(f) = S(f)H(f)$$

Or,

$$R(f) = \frac{1}{2}[U(f - f_c) + U^*(-f - f_c)][C(f - f_c) + C^*(-f - f_c)]$$

where $s(t)$ is a narrowband signal and $h(t)$ is the impulse response of a narrowband system

$$U(f - f_c) \approx 0 \; for \; f < 0 \; and \; C(f - f_c) = 0 \; for \; f < 0.$$

So,

$$U(f - f_c)\, C^*(-f - f_c) = 0$$

and,

$$U^*(-f - f_c)\, C(f - f_c) = 0$$

So,

$$\begin{aligned} R(f) &= \frac{1}{2}[U(f - f_c)C(f - f_c) + U^*(-f - f_c)C^*(-f - f_c)] \\ &= \frac{1}{2}[V(f - f_c) + V^*(-f - f_c)] \end{aligned}$$

where $V(f) = U(f)C(f)$ is the output spectrum of the equivalent low-pass system excited by the equivalent low-pass signal.
So,

$$v(t) = u(t) * c(t)$$

or,

$$v(t) = \int_{-\infty}^{\infty} u(\tau)c(t - \tau)d\tau$$

These relationships between bandpass and equivalent low-pass signals allow us to ignore any linear frequency translations encountered in the modulation of a signal for the purpose of matching its spectral content to the frequency allocation of a particular channel.

25.4 Representations of Bandpass Stationary Stochastic Processes

Let $n(t)$ be a WSS stochastic process with zero mean.

$$n(t) = a(t)\cos[2\pi f_c t + \theta(t)]$$

$$= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

$$= Re[z(t)e^{i2\pi f_c t}]$$

$$a(t) = envelope$$

$$z(t) = x(t) + iy(t) \text{ (complex envelope)}$$

$$E[n(t)] = 0 \Longrightarrow E[x(t)] = E[y(t)] = 0$$

Claim

$$R_X(\tau) = R_Y(\tau)$$

$$R_{XY}(\tau) = -R_{YX}(\tau)$$

Proof

$$R_n(\tau) = E[n(t)n(t-\tau)]$$

$$= E[(x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t)(x(t-\tau)\cos 2\pi f_c(t-\tau) - y(t-\tau)\sin 2\pi f_c(t-\tau))]$$

$$= R_X(\tau)\cos 2\pi f_c t\cos 2\pi f_c(t-\tau)$$

$$+R_Y(\tau)\sin 2\pi f_c t\sin 2\pi f_c(t-\tau)$$

$$-R_{YX}(\tau)\sin 2\pi f_c t\cos 2\pi f_c(t-\tau)$$

$$-R_{XY}(\tau)\cos 2\pi f_c t\sin 2\pi f_c(t-\tau)$$

Use

$$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$R_n(\tau) = \frac{1}{2}[R_X(\tau) + R_Y(\tau)]\cos 2\pi f_c\tau$$

$$+\frac{1}{2}[R_X(\tau) - R_Y(\tau)]\cos 2\pi f_c(2t-\tau)$$

$$-\frac{1}{2}[R_{YX}(\tau) + R_{XY}(\tau)]\sin 2\pi f_c\tau$$

$$-\frac{1}{2}[R_{YX}(\tau) + R_{XY}(\tau)]\sin 2\pi f_c(2t-\tau)$$

RHS must be independent of t for $n(t)$ to be WSS.

$$\implies \quad R_X(\tau) = R_Y(\tau)$$

$$R_{XY}(\tau) = -R_{YX}(\tau)$$

Thus,

$$R_n(\tau) = R_X(\tau)\cos 2\pi f_c\tau - R_{YX}(\tau)\sin 2\pi f_c\tau$$

The autocorrelation function of the equivalent low-pass process

$$z(t) = x(t) + iy(t)$$

is defined as

$$R_Z(\tau) = \frac{1}{2}E[z(t)z^*(t+\tau)]$$

$$= \frac{1}{2}[R_X(\tau) + R_Y(\tau) - iR_{XY}(\tau) + iR_{YX}(\tau)]$$

$$= R_X(\tau) + iR_{YX}(\tau)$$

So,

$$R_n(\tau) = Re[R_z(\tau)e^{i2\pi f_c\tau}]$$

Thus, the autocorrelation function $R_n(\tau)$ of the bandpass stochastic process is determined from $R_Z(\tau)$, the autocorrelation function of the equivalent low-pass process $z(t)$ and the carrier frequency f_c.

Now,

$$S_n(f) = \int_{-\infty}^{\infty}\{Re[R_Z(\tau)e^{i2\pi f_c\tau}]\}e^{-i2\pi f_c\tau}d\tau$$

$$= \frac{1}{2}[S_Z(f-f_c) + S_Z(-f-f_c)]$$

25.4.1 Properties of the In-Phase and Quadrature Components

Since

$$R_{XY}(\tau) = -R_{YX}(\tau)$$

and

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

We get

$$R_{XY}(\tau) = -R_{XY}(-\tau)$$

$$\Longrightarrow R_{XY}(\tau) \textit{ is an odd function of } \tau$$

So,

$$R_{XY}(0) = 0 \Longrightarrow x(t) \textit{ and } y(t) \textit{ are uncorrelated } \underline{for\ \tau = 0.}$$

If $n(t)$ is a Gaussian process, then $x(t+\tau)$ and $y(t)$ are jointly Gaussian and for $\tau = 0$ they are uncorrelated $\Longrightarrow$ independent.
So, in this case their joint pdf is

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

where $\sigma^2 = R_X(0) = R_Y(0) = R_n(0)$

25.4.2 Representation of White Noise

The noise resulting from passing white noise through a spectrally flat (ideal) bandpass filter is termed bandpass white noise.
The equivalent low-pass noise $z(t)$ has

$$S_Z(f) = \begin{cases} N_o, |f| \leq \dfrac{B}{2} \\ 0, |f| > \frac{B}{2}. \end{cases}$$

$$\Longrightarrow R_Z(\tau) = N_o \frac{\sin \pi B \tau}{\pi \tau}$$

As $B \to \infty$

$$R_Z(\tau) \longrightarrow N_o \delta(\tau)$$

The power spectral density for white noise and bandpass white noise is symmetric about $f = 0$, so $R_{YX}(\tau) = 0 \quad \forall\, \tau$.
Thus,

$$R_Z(\tau) = R_X(\tau) = R_Y(\tau)$$

$\Longrightarrow x(t)$ and $y(t)$ are uncorrelated for all time shifts τ and the autocorrelation functions of $z(t), x(t)$ and $y(t)$ are all equal.