24.0 LMMSE Estimation and K-L Expansions

Let

$$X(u,t) = S(u,t) + W(u,t)$$

where $S(u,t)$ is a mean zero Gaussian process and $W(u,t)$ is mean zero white noise. Furthermore, $S(u,t)$ is independent of $W(u,t)$. We observe $X(u,t)$ in the interval $[0,T]$. We want to estimate S so that

$$E\left[\int_0^T |S(u,t) - \hat{S}(u,t)|^2 dt\right]$$

is minimized. Write

$$S(u,t) = \sum_{n=1}^{\infty} S_n(u)\phi_N(t)$$

where $\phi_N(t)$ is a complete set of orthonormal eigenfunctions. Also,

$$W(u,t) = \sum_{n=1}^{\infty} W_n(u)\phi_n(t).$$

So

$$X(u,t) = \sum_{n=1}^{\infty} X_n(u)\phi_n(t)$$

where

$$X_n(u) = S_n(u) + W_n(u).$$

We know

$$\hat{S}_n(u) = \frac{\sigma_{S_n}^2}{\sigma_{S_n}^2 + \sigma_W^2} X_n(u).$$

Thus

$$\hat{S}(u,t) = \sum_{n=1}^{\infty} \hat{S}_n(u)\phi_n(t).$$

Note

$$E\left[|S_n(u) - \hat{S}_n(u)|^2\right] = \frac{\sigma_{S_n}^2 \sigma_W^2}{\sigma_{S_n}^2 + \sigma_W^2}$$

and the total mean square error is

$$MSE = \sum_{n=1}^{\infty} \frac{\sigma_{S_n}^2 \sigma_W^2}{\sigma_{S_n}^2 + \sigma_W^2}.$$