23.0 Stochastic Differential Equations

The equation

$$\frac{dY(t)}{dt} = X(t) \text{ (M.S.)}$$

which implies

$$E\left[\left|\frac{dY(t)}{dt} = X(t)\right|^2\right] = 0$$

has solution

$$Y(t) = \int_{t_0}^{t} X(s)ds + Y(t_0), \quad t \geq t_0.$$

More generally,

$$\sum_{\ell=0}^{n} a_\ell Y^{(\ell)}(t) = X(t) \qquad (*)$$

where,

$$Y^{(\ell)}(t) = \frac{d^\ell Y(t)}{dt^\ell}$$

with initial condition $Y^{(\ell)}(t_0) = Y_\ell$.

Now

$$E\left[Y^{(\ell)}(t)\right] = E\left[\frac{d^\ell Y(t)}{dt^\ell}\right] = \frac{d^\ell E\left[Y(t)\right]}{dt^\ell}.$$

So,

$$\sum_{\ell=0}^{n} a_\ell \frac{d^\ell \mu_Y(t)}{dt^\ell} = \mu_X(t)$$

with initial conditions

$$\left.\frac{d^\ell \mu_Y(t)}{dt^\ell}\right|_{t=t_0} = E\left[Y_\ell\right].$$

Now in $(*)$ multiple by $X^*(t_2)$ to get

$$E\left[X^*(t_2)\sum_{\ell=0}^{n} a_\ell Y^{(\ell)}(t_1)\right] = E\left[X^*(t_2)X(t_1)\right]$$

or

$$\sum_{\ell=0}^{n} a_\ell \frac{\partial^\ell R_{YX}(t_1,t_2)}{\partial t_1^\ell} = R_X(t_1,t_2)$$

with initial conditions

$$\left.\frac{\partial^\ell R_{YX}(t_1,t_2)}{\partial t_1^\ell}\right|_{t_1=t_0} = E\left[Y_\ell X^*(t_2)\right],\ \ell = 0,1,\ldots,n-1.$$

In $(*)$ multiply by $Y^*(t_2)$ and take expectations to get

$$\sum_{\ell=0}^{n} a_\ell \frac{\partial^\ell R_Y(t_1,t_2)}{\partial t_1^\ell} = R_{XY}(t_1,t_2)$$

with initial conditions

$$\left.\frac{\partial^\ell R_Y(t_1,t_2)}{\partial t_1^\ell}\right|_{t_1=t_0} = E\left[Y_\ell Y^*(t_2)\right],\ \ell = 0,1,\ldots,n-1.$$

Example: Let $X(t)$ be a stationary random process with mean μ_X and covariance $K_X(\tau) = \sigma^2\delta(\tau)$. Let $Y(t)$ be the mean square solution to the stochastic differential equation

$$\frac{dY(t)}{dt} + \beta Y(t) = X(t), \quad t \geq 0$$

with initial condition $Y(0) = 0$ and $\beta > 0$. Then taking expectations of both sides we get

$$\frac{d\mu_Y(t)}{dt} + \beta\mu_Y(t) = \mu_X$$

with initial condition $\mu_Y(0) = E[Y(0)] = 0$. Taking Laplace transforms we get

$$s\mu_Y(s) - \mu_Y(0) + \beta\mu_Y(s) = \frac{\mu_X}{s}$$

or

$$\mu_Y(s) = \frac{\mu_X}{s(s+\beta)} = \mu_X\left(\frac{1/\beta}{s} - \frac{1/\beta}{s+\beta}\right) = \frac{\mu_X}{\beta}\left(\frac{1}{s} - \frac{1}{s+\beta}\right).$$

Taking inverse Laplace transforms we get

$$\mu_Y(t) = \frac{\mu_X}{\beta}\left[1 - e^{-\beta t}\right], \quad t \geq 0.$$

Now let us rewrite our original equation using the variable t_1 instead of t:

$$\frac{dY(t_1)}{dt_1} + \beta Y(t_1) = X(t_1), \quad t_1 \geq 0.$$

Since

$$\frac{d\mu_Y(t_1)}{dt_1} + \beta\mu_Y(t_1) = \mu_X$$

we can write

$$\frac{d(Y(t_1) - \mu_Y(t_1))}{dt_1} + \beta(Y(t_1) - \mu_Y(t_1)) = (X(t_1) - \mu_X), \quad t_1 \geq 0.$$

Multiplying by $(X^*(t_2) - \mu_X^*)$ we get

$$\frac{d\,(Y(t_1) - \mu_Y(t_1))\,(X^*(t_2) - \mu_X^*)}{dt_1} + \beta\,(Y(t_1) - \mu_Y(t_1))\,(X^*(t_2) - \mu_X^*)$$

$$= (X(t_1) - \mu_X)\,(X^*(t_2) - \mu_X^*)\,.$$

Taking expectations this becomes

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + \beta K_{YX}(t_1, t_2) = \sigma^2\delta(t_1 - t_2)$$

with initial condition $K_{YX}(0, t_2) = 0$.

For $t_1 < t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + \beta K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(0, t_2) = 0$ which implies $K_{YX}(t_1, t_2) = 0$.

For $t_1 \geq t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + \beta K_{YX}(t_1, t_2) = \sigma^2\delta(t_1 - t_2)$$

due to the jump of σ^2 at $t_1 = t_2$. Taking Laplace transforms we get

$$s_1 K_{YX}(s_1, t_2) + \beta K_{YX}(s_1, t_2) = \sigma^2 e^{-s_1 t_2}$$

so

$$K_{YX}(s_1, t_2) = \frac{\sigma^2}{s_1 + \beta} e^{-s_1 t_2}.$$

Hence

$$K_{YX}(t_1, t_2) = \sigma^2 e^{-\beta(t_1 - t_2)}$$

for $t_1 \geq t_2$ and is zero otherwise.

We will now derive this last result another way. Instead of considering $t_1 \geq t_2$ we will consider $t_1 > t_2$ with initial condition at $t_1 = t_2$.

For $t_1 > t_2$ we get

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + \beta K_{YX}(t_1, t_2) = 0.$$

Taking Laplace transforms we get

$$s_1 K_{YX}(s_1, t_2) - K_{YX}(t_2, t_2) e^{-s_1 t_2} + \beta K_{YX}(s_1, t_2) = 0$$

or

$$s_1 K_{YX}(s_1, t_2) - \sigma^2 e^{-s_1 t_2} + \beta K_{YX}(s_1, t_2) = 0$$

so

$$K_{YX}(s_1, t_2) = \frac{\sigma^2}{s_1 + \beta} e^{-s_1 t_2}.$$

Hence

$$K_{YX}(t_1, t_2) = \sigma^2 e^{-\beta(t_1 - t_2)}$$

for $t_1 \geq t_2$ and is zero otherwise. This is the same answer as before.

Repeating the above procedure but now multiplying by $(Y^*(t_2) - \mu_Y^*(t_2))$ we get,

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + \beta K_Y(t_1, t_2) = K_{XY}(t_1, t_2)$$

with initial condition $K_Y(0, t_2) = 0$, or

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + \beta K_Y(t_1, t_2) = K_{YX}(t_2, t_1)$$

$$= \sigma^2 e^{-\beta(t_2 - t_1)}$$

for $t_2 \geq t_1$ and is zero otherwise.

So for $0 < t_1 \leq t_2$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + \beta K_Y(t_1, t_2) = \sigma^2 e^{-\beta(t_2 - t_1)}.$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1, t_2) + \beta K_Y(s_1, t_2) = \frac{\sigma^2 e^{-\beta t_2}}{s_1 - \beta}$$

or

$$K_Y(s_1, t_2) = \frac{\sigma^2 e^{-\beta t_2}}{(s_1 - \beta)(s_1 + \beta)} = \frac{\sigma^2 e^{-\beta t_2}/2\beta}{(s_1 - \beta)} - \frac{\sigma^2 e^{-\beta t_2}/2\beta}{s_1 + \beta}$$

so

$$K_Y(t_1, t_2) = \frac{\sigma^2}{2\beta} e^{-\beta t_2} \left(e^{\beta t_1} - e^{-\beta t_1} \right).$$

For $t_1 > t_2$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + \beta K_Y(t_1, t_2) = 0.$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1, t_2) - K_Y(t_2, t_2) e^{-s_1 t_2} + \beta K_Y(s_1, t_2) = 0$$

or

$$s_1 K_Y(s_1, t_2) - \frac{\sigma^2}{2\beta} \left(1 - e^{-2\beta t_2} \right) e^{-s_1 t_2} + \beta K_Y(s_1, t_2) = 0$$

so

$$K_Y(s_1, t_2) = \frac{\sigma^2}{2\beta} \frac{\left(1 - e^{-2\beta t_2} \right)}{s_1 + \beta} e^{-s_1 t_2}$$

So taking the inverse tranform we get

$$K_Y(t_1, t_2) = \frac{\sigma^2}{2\beta} \left(1 - e^{-2\beta t_2} \right) e^{-\beta(t_1 - t_2)}.$$

Now let $t_2 = t,\ t_1 = t + \tau$. Then

$$K_Y(t + \tau, t) = \begin{cases} \frac{\sigma^2}{2\beta} \left(1 - e^{-2\beta t} \right) e^{-\beta\tau}, & \tau > 0, \\ \\ \frac{\sigma^2}{2\beta} e^{-\beta t} \left(e^{\beta(t+\tau)} - e^{-\beta(t+\tau)} \right), & \tau \leq 0. \end{cases}$$

$$\frac{\partial K_Y(t_1, t_2)}{\partial t_1} + \beta K_Y(t_1, t_2) = \frac{\sigma^2}{2\beta} e^{-\beta t_2} \left(e^{\beta t_1} - e^{-\beta t_1} \right) \Big|_{t_1 = t_2} \delta(t_1 - t_2)$$

which becomes

$$\frac{\partial K_Y(t_1,t_2)}{\partial t_1} + \beta K_Y(t_1,t_2) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t_2}\right)\delta(t_1 - t_2).$$

Taking Laplace transforms we get

$$s_1 K_Y(s_1,t_2) + \beta K_Y(s_1,t_2) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t_2}\right)e^{-s_1 t_2}$$

or

$$K_Y(s_1,t_2) = \frac{\sigma^2}{2\beta}\frac{\left(1 - e^{-2\beta t_2}\right)}{s_1 + \beta}e^{-s_1 t_2}$$

So taking the inverse tranform we get

$$K_Y(t_1,t_2) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t_2}\right)e^{-\beta(t_1 - t_2)}.$$

Now let $t_2 = t,\ t_1 = t + \tau$. Then

$$K_Y(t+\tau,t) = \begin{cases} \dfrac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)e^{-\beta\tau}, & \tau > 0, \\ \\ \dfrac{\sigma^2}{2\beta}e^{-\beta t}\left(e^{\beta(t+\tau)} - e^{-\beta(t+\tau)}\right), & \tau \leq 0. \end{cases}$$

Now let $t \to \infty$ to get

$$K_Y(\tau) = \frac{\sigma^2}{2\beta}e^{-\beta|\tau|}.$$

Thus $Y(t)$ is asymptotically WSS.