22.0 Karhunen-Loeve Expansion

We will assume zero mean in this section.

Mercer's Theorem: If $K(t, s)$ is a continuous positive semi-definite function, then if $\{\lambda_n, \phi_n(t)\}$ are a complete set of orthonormal solutions to

$$\int_a^b K(t,s)\phi_n(s)ds = \lambda_n \phi_n(t), \quad a \le t \le b$$

then

$$K(t,s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t)\phi_n^*(s) \ \ \forall \ t, s \in [a, b].$$

Here $\lambda_n = E[|X_n|^2]$.

Note for vectors we have

$$K_X = \sum_{n=1}^{N} \lambda_i e_i e_i^{*T} = E\Lambda E^{*T}.$$

Note that if $a = E^{*T}X$ then $a_1, \ldots, a_N$ are uncorrelated random variables with $E[a_i^2] = \lambda_i$.

K-L Theorem: If $X(t)$ is a zero-mean 2nd order random process on $[-T/2, T/2]$ with continuous covariance $K_X(t_1, t_2)$ then

$$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t), \quad \forall \ |t| \le T/2 \quad (1)$$

where

$$X_n = \int_{-T/2}^{T/2} X(t)\phi_n^*(t)dt \quad (2)$$

and $\{\phi_n(t)\}$ are an orthonormal set of solutions to

$$\int_{-T/2}^{T/2} K_X(t_1, t_2)\phi_n(t_2)dt_2 = \lambda_n \phi_n(t_1), \ \forall \ |t_1| \le T/2 \quad (3)$$

and

$$E[X_n X_m^*] = \lambda_n \delta(n - m). \quad (4)$$

Proof:
$1, 2, 4 \Rightarrow 3.$
By (1)

$$E[X(t_1)X_n^*] = \sum_{n=1}^{\infty} E[X_m X_n^*]\phi_m(t_1)\ (MS)$$

and by (4)

$$= E[|X_n|^2]\phi_n(t_1) = \lambda_n \phi_n(t_1)$$

and by (2)

$$= E[X(t_1)\int_{-T/2}^{T/2} X^*(t_2)\phi_n(t_2)dt_2].$$

So

$$\lambda_n \phi_n(t) = \int_{-T/2}^{T/2} K_X(t_1, t_2)\phi_n(t_2)dt_2.$$

$1, 2, 3 \Rightarrow 4.$

$$E[X(t)X_m^*] = \lambda_m \phi_m(t).$$

$$E[X_n X_m^*] = E\left[\int_{-T/2}^{T/2} X(t)\phi_n^*(t)dt \cdot X_m^*\right]$$

$$= \int_{-T/2}^{T/2} \lambda_m \phi_m(t)\phi_n^*(t)dt$$

$$= \lambda_n \delta(n-m).$$

Now let

$$\hat{X}(t) = \sum_{n=1}^{\infty} X_n \phi_n(t)$$

where

$$X_n = \int_{-T/2}^{T/2} X(t)\phi_n^*(t)dt.$$

Then

$$E[|X(t) - \hat{X}(t)|^2] = 0.$$

So

$$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \quad (MS).$$

Example: White Noise.

$$K_X(t_1, t_2) = \sigma^2 \delta(t_1 - t_2).$$

$$\int_{-T/2}^{T/2} \sigma^2 \delta(t_1 - t_2)\phi(t_2)dt_2 = \lambda\phi(t_1).$$

$$\sigma^2 \phi(t_1) = \lambda\phi(t_1)$$

so

$$\lambda = \sigma^2.$$

Therefore, a basis for white noise $\{\phi_n(t)\}$ can be any complete set of orthonormal functions with corresponding eigenvalues $\lambda_n = \sigma^2$.

Example: Bandlimited Noise. Assume zero-mean and WSS.

$$S_X(f) = \int_{-\infty}^{\infty} K_X(\tau)\exp(-i2\pi f\tau)d\tau.$$

$X(t)$ is bandlimited if $S_X(f) = 0$ for $|f| > f_{max}$. Suppose $S_X(f) = 1$ for $|f| \leq f_{max}$. Then

$$R(\tau) = \int_{-f_{max}}^{f_{max}} 1 \cdot \exp(i2\pi f\tau)df = \frac{\sin 2\pi f_{max}\tau}{\pi\tau}.$$

$$\int_{-T/2}^{T/2} \frac{\sin 2\pi f_{max}(t_1 - t_2)}{\pi(t_1 - t_2)}\phi_n(t_2)dt_2 = \lambda_n\phi_n(t_1).$$

Prolate spheroidal functions solve this equation for ϕ_n.