21.0 Ergodicity

21.1 Ergodic in Mean

Definition: The WSS RP $X(t)$ is said to be *ergodic in mean* if

$$\mu_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt \to \mu_X \text{ in M.S. as } T \to \infty.$$

$$E\left[\mu_T\right] = \frac{1}{2T}\int_{-T}^{T} E\left[X(t)\right]dt = \mu_x$$

so μ_T is an unbiased estimator for μ_X.

$$Var\left[\mu_T\right] = \left(\frac{1}{2T}\right)^2 \int_{-T}^{T}\int_{-T}^{T} K_X(t_1 - t_2)dt_1 dt_2.$$

The WSS random process $X(t)$ is ergodic in mean if and only if

$$\lim_{T\to\infty} \left(\frac{1}{2T}\right)^2 \int_{-T}^{T}\int_{-T}^{T} K_X(t_1 - t_2)dt_1 dt_2 = 0.$$

By letting $s = t_1 + t_2$ and $\tau = t_1 - t_2$ we can write

$$Var\left[\mu_T\right] = \left(\frac{1}{2T}\right)^2 \int_{-2T}^{2T}\int_{-2T-|\tau|}^{2T-|\tau|} K_X(\tau)dsd\tau.$$

Simplifying this expression leads to the following theorem:

Theorem: A WSS random process $X(t)$ is ergodic in the mean if and only if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T} K_X(\tau)\left[1 - \frac{|\tau|}{2T}\right]d\tau = 0.$$

Proposition: The WSS RP $X(t)$ is ergodic in mean if

$$\int_{-\infty}^{\infty} |K_X(\tau)|dt < \infty.$$

Proof: Note that

$$1 - \frac{|\tau|}{2T} \leq 1 \text{ for } -2T \leq \tau \leq 2T.$$

Thus

$$Var\left[\mu_T\right] \leq \frac{1}{2T}\int_{-2T}^{2T}\left|K_X(\tau)\right|d\tau \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Proposition: The WSS RP $X(t)$ is ergodic in mean if

$$\lim_{\tau\rightarrow\infty}|K_X(\tau)| = 0.$$

Proof: Let

$$K_m = \max_{\tau}\left\{K_X(\tau\right\}.$$

Let $\delta > 0$. Then there exists M such that $|K_X(\tau)| \leq \delta$ for $|\tau| \geq M$. So

$$Var(\mu_T) \leq \frac{1}{2T}\int_{-2T}^{2T}|K_X(\tau)|d\tau$$

$$\leq \frac{1}{2T}\left[\int_{-M}^{M}K_m d\tau + \int_{-2T}^{M}\delta d\tau + \int_{M}^{2T}\delta d\tau\right]$$

(we make $T > M$)

$$= \frac{1}{2T}\left[2MK_m + (4T-2M)\delta\right]$$

$$\leq \frac{M}{T}K_m + 2\delta.$$

For any $\epsilon > 0$ let

$$\delta = \frac{1}{2}\left[\epsilon - \frac{M}{T}K_m\right].$$

$\delta > 0$ as $T \rightarrow \infty$. Here

$$Var(\mu_T) \leq \epsilon.$$

Since ϵ is arbitray, $Var(\mu_T) \rightarrow 0$.

21.2 Ergodic in Correlation

Definition: The WSS RP $X(t)$ is said to be *ergodic in correlation at shift* λ if

$$\lim_{T\rightarrow\infty}\frac{1}{2T}\int_{-T}^{T}X(t+\lambda)X(t)dt = R_X(\lambda) \text{ (M.S.)}.$$

If this holds for all λ we say $X(t)$ is *ergodic in correlation.*

Let

$$\phi_\lambda(t) = X(t+\tau)X(t).$$

Theorem: The WSS RP X(t) is ergodic in correlation if and only if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} K_{\phi_\lambda}(\tau) \left[1 - \frac{|\tau|}{2T}\right] d\tau = 0.$$

Proof: The proof is the same as proving ergodic in mean for $\phi_\lambda(t)$.

21.3 Ergodic in Mean Square

Definition: The WSS RP $X(t)$ is said to be *ergodic in mean square* if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)dt = R_X(0) \text{ (M.S.)}.$$

Example: Let

$$X(t) = A\cos[2\pi f_0 t + \theta]$$

where $A \sim N(0,1), \quad \theta \sim U[-\pi, \pi]$. Then

$$E[X(t)] = 0.$$

$$K_X(t+\tau, t) = E[X(t+\tau)X(t)] = \frac{1}{2\pi}\cos(2\pi f_0 \tau)$$

which implies $X(t)$ is WSS.

Is $X(t)$ ergodic in mean? Note that

$$\int_{-\infty}^{\infty} |K_X(\tau)| d\tau = \infty$$

so we need to check the more strict condition

$$Var(\mu_T) = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) \frac{1}{2}\cos(2\pi f_0 \tau) d\tau.$$

Define

$$tri(2T) = \left(1 - \frac{|\tau|}{2T}\right).$$

Then $tri(2T)$ is the convolution of two rectangles each with height $1/2T$ from $-T$ to T. Hence

$$Var(\mu_T) = \lim_{T\to\infty} \frac{1}{4T} Re\left\{\int_{-\infty}^{\infty} tri(2T)\exp(-i2\pi f_0 \tau) d\tau\right\}$$

$$= \frac{1}{4T} Re \left\{ F[tri(2T)|_{f=f_0} \right\}$$

where F denotes a Fourier transform. Now the Fourier transform of $tri(2T)$ is

$$2T \left(\frac{\sin 2\pi f T}{2\pi f T} \right)^2$$

so $Var(\mu_T) \to 0$ as $T \to \infty$. Therefore $X(t)$ is ergodic in mean.

Is $X(t)$ ergodic in M.S.?

$$\frac{1}{2T} \int_{-T}^{T} X^2(t)dt = \frac{1}{2T} \int_{-T}^{T} A^2 \cos^2(2\pi f_0 t + \theta)dt$$

$$= \frac{A^2}{2T} \int_{-T}^{T} \cos^2(2\pi f_0 t + \theta)dt$$

which yields a different result for different A since A is random. Hence $X(t)$ is not ergodic in M.S. and so is not ergodic in correlation either.

21.4 Ergodic in Distribution

Suppose $X(t)$ is stationary in first and second order distributions. Then

$$F_X(x;t) = P\left[X(u,t) \le x\right] = F_X(x;0) \ \forall \ t \in T,$$

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_1 + \tau, t_2 + \tau) \ \forall \ t_1, t_2, \tau \in T.$$

Here we estimate $F_X(x;0)$ from a single sample function. Let

$$I_x(t) = \begin{cases} 1, & X(t) \le x, \\ 0, & \text{elsewhere.} \end{cases}$$

$I_x(t)$ is the indicator function. Let

$$\hat{F}_X(x) = \frac{1}{2T} \int_{-T}^{T} I_x(t)dt.$$

Then

$$E\left[\hat{F}_X(x)\right] = E\left[\frac{1}{2T} \int_{-T}^{T} I_x(t)dt\right] = \frac{1}{2T} \int_{-T}^{T} E\left[I_x(t)\right]dt.$$

Now

$$E[I_x(t)] = P[X(t) \leq x] = F_X(x;t) = F_X(x;0)$$

where the last equality follows from first order stationarity. Thus

$$E\left[\hat{F}_X(x)\right] = F_X(x;0).$$

Now consider

$$E[I_x(t_1)I_x(t_2)] = P[X(t_1) \leq x, X(t_2) \leq x]$$

$$= F_X(x,x;t_1,t_2) = F_X(x,x;t_1-t_2,0).$$

So $I_x(t)$ will be a WSS RP provided $X(t)$ is stationary of order 2. Applying our previous theorem we get the following theorem.

Theorem: The WSS RP X(t) which is stationary up to order 2 is ergodic in distribution if and only if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K_{I_x}(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau = 0$$

where

$$K_{I_x}(\tau) = E\left[\left(I_x(t) - E[I_x(t)]\right)\left(I_x(t-\tau) - E[I_x(t-\tau)]\right)\right]$$

$$= F_X(x,x;\tau) - F_X^2(x).$$