20.0 Mean Square Calculus

20.1 Mean Square Continuity

Imposing continuity on all sample functions is too restrictive in general, therefore we use mean square continuity.

Definition: The random process $X(t) = X(u,t)$ is *mean square continuous at* $t = t_0$ if

$$\lim_{t \to t_0} E\left[|X(t) - X(t_0)|^2\right] = 0.$$

If it holds for all $t_0 \in T$, $X(t)$ is *mean square continuous.*

Theorem: $X(t)$ is mean square continuous at t_0 if and only if $R_X(t_1, t_2)$ is continuous at the point $t_1 = t_2 = t_0$.

Proof: "$\Leftarrow$".

$$E\left[|X(t) - X(t_0)|^2\right] = R_X(t,t) - R_x(t_0,t) - R_x(t,t_0) + R_X(t_0,t_0) = 0$$

as $t \to t_0$.

"$\Leftarrow$". If the $LHS \to 0$ then the $RHS \to 0$ so $R_X(t_1, t_2)$ must be continuous at $t_1 = t_2 = t_0$ as $t \to t_0$.

Note: If $X(t)$ is WSS then $X(t)$ is mean square continuous if and only if $R_X(\tau)$ is continuous at $\tau = 0$.

Theorem: The mean function of a mean square continuous random process is a continuous function.

Proof:

$$(\mu_X(t) - \mu_X(t_0))^2 = [E(X(t) - X(t_0))]^2 \le E\left[(X(t) - X(t_0))^2\right].$$

As $t \to t_0$ the $RHS = 0 \Rightarrow LHS = 0$

$$\Rightarrow \lim_{t \to t_0} (\mu_X(t) - \mu_X(t_0))^2 = 0$$

so

$$\mu_X(t) \to \mu_X(t_0)$$

i.e.,

$$\lim_{t \to t_0} E\left[X(t)\right] = E\left[\lim_{t \to t_0} X(t)\right].$$

Example: Wiener process.

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2).$$

As

$$(t_1, t_2) \to (t, t)$$

we have

$$\alpha \min(t_1, t_2) \to \alpha t.$$

Is this M.S. continuous? Consider

$$|\alpha \min(t_1, t_2) - t|.$$

Let $t_1 = t + \epsilon_1,\ \epsilon_1 > 0, \quad t_2 = t + \epsilon_2,\ \epsilon_2 > 0$. Then

$$\alpha |\min(t + \epsilon_1, t + \epsilon_2) - t| \le \alpha \max(\epsilon_1, \epsilon_2) \to 0 \text{ as } \epsilon_1, \epsilon_2 \to 0$$

$\Rightarrow R_X(t_1, t_2)$ is continuous at this point so the Wiener process is M.S. continuous.

20.2 Mean Square Derivatives

Theorem: The M.S. derivative exists at t if and only if

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$$

exists when $(t_1, t_2) = (t, t)$.

Proof: Since $X'(t)$ is not known we will use the Cauchy criterion.

$$\lim_{\epsilon_1, \epsilon_2 \to 0} E\left[\left|\frac{X(t+\epsilon_1) - X(t)}{\epsilon_1} - \frac{X(t+\epsilon_2) - X(t)}{\epsilon_2}\right|^2\right]$$

$$= \lim_{\epsilon_1,\epsilon_2 \to 0} E\left[\left|\frac{X(t+\epsilon_1)-X(t)}{\epsilon_1}\right|^2\right] + E\left[\left|\frac{X(t+\epsilon_2)-X(t)}{\epsilon_2}\right|^2\right]$$

$$-2E\left[\left(\frac{X(t+\epsilon_1)-X(t)}{\epsilon_1}\right)\left(\frac{X^*(t+\epsilon_2)-X^*(t)}{\epsilon_2}\right)\right].$$

The first two terms of this last expression are of the form

$$E\left[\left|\frac{X(t+\epsilon)-X(t)}{\epsilon}\right|^2\right] = \frac{1}{\epsilon}\left[\frac{R_X(t+\epsilon,t+\epsilon)-R_X(t,t+\epsilon)}{\epsilon} - \frac{R_X(t+\epsilon,t)-R_X(t,t)}{\epsilon}\right].$$

This converges to

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)$$

if the second mixed partial derivative w.r.t. t_1 and t_2 exists at the point (t,t). The last term in our expression above is

$$-2E\left[\left(\frac{X(t+\epsilon_1)-X(t)}{\epsilon_1}\right)\left(\frac{X^*(t+\epsilon_2)-X^*(t)}{\epsilon_2}\right)\right]$$

$$= \frac{-2}{\epsilon_1}\left[\frac{R_X(t+\epsilon_1,t+\epsilon_2)-R_X(t+\epsilon_1,t)}{\epsilon_2} - \frac{R_X(t,t+\epsilon_2)-R_X(t,t)}{\epsilon_2}\right]$$

which converges to

$$-2\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t).$$

So our expression becomes

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t) + \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t) - 2\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t) = 0.$$

Thus the M.S. derivative exists.

Also, the M.S. derivative of a W.S. stationary R.P. $X(t)$ exists if $R_X(\tau)$ has derivatives up to order two at $\tau = 0$.

Example: Let

$$X(u,t) = A(u)\cos(2\pi t).$$

Then

$$R_X(t_1,t_2) = E[A^2(u)\cos(2\pi t_1)\cos(2\pi t_2)].$$

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)\bigg|_{t_1=t_2=t} = 4\pi^2 E[A^2(u)] \sin^2(2\pi t)$$

$\Rightarrow X(t)$ has a M.S. derivative at every point t.

Example:

$$R_X(\tau) = \sigma^2 \exp(-\alpha^2\tau^2).$$

$$\frac{d}{d\tau} R_X(\tau) = -2\alpha^2\sigma^2\tau \exp(-\alpha^2\tau^2).$$

$$\frac{d^2}{d\tau^2} R_X(\tau) = 4\alpha^4\sigma^2\tau \exp(-\alpha^2\tau^2) - 2\alpha^2\sigma^2 \exp(-\alpha^2\tau^2).$$

$$R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau) = (1 - 2\alpha^2\tau^2) 2\alpha^2\sigma^2 \exp(-\alpha^2\tau^2).$$

Example: Wiener process.

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2).$$

$$\frac{\partial R_X(t_1, t_2)}{\partial t_1} = \alpha u(t_2 - t_1).$$

But the derivative of a step function does not exist at its point of discontinuity, so the second mixed partial derivative does not exist. Hence the Wiener process does not have a M.S. derivative at any point. Instead we use a generalized derivative as

$$\frac{\partial R_X(t_1, t_2)}{\partial t_1 \partial t_2} = \alpha\delta(t_2 - t_1)$$

$$R_{X'}(t_1, t_2) = \alpha\delta(t_2 - t_1)$$

or

$$R_{X'}(\tau) = \alpha\delta(\tau)$$

which implies a flat spectral density. So the generalized M.S. derivative of the Wiener process is white Gaussian noise.

20.3 Mean Square Integrals

Example: White Gaussian noise.

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sigma^2 \delta(t_1 - t_2).$$

Let

$$Y(t) = \int_0^t X(\tau) d\tau.$$

$Y(t)$ is a stochastic process since for a fixed $t = t_0$, $Y(t_0)$ is a random variable.

$$E[Y(t)] = 0.$$

$$R_Y(t_1, t_2) = E[Y(t_1)Y^*(t_2)] = K_Y(t_1, t_2).$$

$$\begin{aligned}
K_Y(t_1, t_2) &= \int_0^t \int_0^t K_X(s_1, s_2) ds_1 ds_2 \\
&= \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(s_1 - s_2) ds_1 ds_2 \\
&= \sigma^2 \int_0^{t_2} u(t_1 - s_2) ds_2 \\
&= \sigma^2 \int_0^{\min(t_1, t_2)} ds_2 \\
&= \sigma^2 \min(t_1, t_2)
\end{aligned}$$

which implies a Wiener process. Note that $Y(t)$ is Gaussian also. Thus, the M.S. integral of white noise is the Wiener process.