19.0 Wiener Process

Let $z(n)$ be a Bernoulli sequence where

$$P(z(n) = s) = P(z(n) = -s) = 1/2$$

where s is the step size. We have

$$E[z(n)] = 0, \quad E[z^2(n)] = s^2.$$

Let

$$x(n) = \sum_{k=1}^{n} z(k).$$

Then, $x(n)$ is a discrete random walk. Now

$$P\left(x(n) = rs\right) = P(k \text{ successes and } (n-k) \text{ failures in } n \text{ trials})$$

where,

$$rs = ks - (n-k)s$$

i.e.,

$$k = \frac{r+n}{2}, \quad \text{an integer.}$$

So

$$P\left(x(n) = rs\right) = P\left(\frac{r+n}{2} \text{ successes in } n \text{ trials}\right)$$

$$= \binom{n}{\frac{n+r}{2}} 2^{-n}, \quad \frac{n+r}{2} \text{ an integer.}$$

Now

$$E[x(n)] = E\left[\sum_{k=1}^{n} z(k)\right] = 0$$

$$Var[x(n)] = E[x^2(n)] = E\left[\left(\sum_{k=1}^{n} z(k)\right)^2\right] = n \cdot Var[z(k)] = ns^2.$$

Define the RP with index set $T = [0, \infty)$ as follows:

$$X_\tau(t) = \sum_{k=1}^{\infty} z(k) u(t - k\tau)$$

where, τ is a time interval, s is a step size.

The Wiener process is formed by letting $\tau, s \to 0$ so that the limit is a continuous time RP (sample paths are continuous) and the variance is not trivial (i.e., $\neq 0, \neq \infty$).

Now

$$E[X_\tau(t)] = 0, \quad E[X_\tau^2(t)|_{t=n\tau}] = ns^2.$$

So

$$E[X_\tau^2(t)] = s^2 t/\tau.$$

Let $s^2 = \alpha\tau$. Then

$$E[X_\tau^2(t)] = \frac{\alpha\tau t}{\tau}.$$

We get

$$\lim_{\tau\to 0} E[X_\tau^2(t)] = \alpha t.$$

Let

$$X(t) = \lim_{\tau\to 0} X_\tau(t).$$

$X(t)$ is a Wiener process.

By the central limit theorem the first order density is $\sim N(0, \alpha t)$. We get

$$f_X(x;t) = \frac{1}{\sqrt{2\pi\alpha t}} \exp\left(-\frac{x^2}{2\alpha t}\right), \ t > 0.$$

For any $0 \leq t' < t$, the increment

$$\Delta = X(t) - X(t')$$

is a random variable having a Gaussian distribution with mean zero and variance $\alpha(t - t')$. Note

$$E[X(t) - X(t')] = 0, \ E[(X(t) - X(t'))^2] = \alpha(t - t'), \ t > t'.$$

The increment is independent of $X(\hat{t}) \ \forall \ \hat{t} \leq t'$.

Covariance

$$K_X(t, s) = E[X(t)X(s)].$$

For $t > s$ compute

$$E[(X(t) - X(s))X(s)] = E[X(t) - X(s)]E[X(s)] = 0.$$

So

$$E[X(t)X(s)] = E[X^2(s)] = \alpha s,\ t > s$$

and similarly

$$E[X(t)X(s)] = E[X^2(t)] = \alpha t,\ t \leq s$$

Thus

$$K_X(t, s) = \alpha \min(t, s).$$

All nth order pdf's are Gaussian. Hence, the Wiener process is a special Gaussian process. It turns out the mean square derivative of a Wiener process is white noise.