

18.0 Wiener Filters

18.1 Discrete Case

$$x(n) \longrightarrow \underbrace{\mathbf{H}(\mathbf{f})}_{\text{causal}} \longrightarrow \underbrace{\oplus}_{\uparrow v(n)} \longrightarrow y(n) \longrightarrow \mathbf{G}(\mathbf{f}) \longrightarrow \hat{x}(n)$$

Find linear $G(f)$ so that $E[|x(n) - \hat{x}(n)|^2]$ is minimized.

Assume $x(n)$ is zero mean, $K_x, K_v, h(n)$ are known, and $E[x(n)v(n)] = 0 \quad \forall n, m$.

We will need

$$\begin{aligned} K_{xy}(n) &= E[x(m)y^*(m-n)] \\ &= E \left[x(m) \left(\sum_{k=0}^{\infty} h^*(k)x^*(m-n-k) + v^*(m-n) \right) \right] \\ K_{xy}(n) &= \sum_k h^*(k)K_x(n+k) \\ \implies [S_{XY}(f) &= H^*(f)S_X(f)] \end{aligned}$$

$$\begin{aligned} K_y(n) &= E[y(m)y^*(m-n)] \\ &= \sum_k \sum_{\ell} h(k)h^*(\ell)K_x(n+\ell-k) + K_v(n) \\ \implies [S_Y(f) &= H(f)S_X(f)H^*(f) + S_v(f)] \end{aligned}$$

$$x(n) \longrightarrow \mathbf{H}(\mathbf{f}) \longrightarrow \underbrace{\oplus}_{\uparrow v(n)} \longrightarrow y(n) \longrightarrow \mathbf{A}(\mathbf{f}) \longrightarrow \underbrace{w(n)}_{\text{white } \sigma^2=1} \longrightarrow \mathbf{B}(\mathbf{f}) \longrightarrow \hat{x}(n)$$

We wish to min $\underbrace{\{E[|x(n) - \hat{x}(n)|^2]\}}_{\text{MSE}}$

$$\begin{aligned}
\mathbf{MSE} &= E[|x(n) - \sum_{k=-\infty}^{\infty} b(k)w(n-k)|^2] \\
&= E[|x(n)|^2] - E[x(n) \sum_k b^*(k)w^*(n-k)] \\
&\quad - E\left[\left(\sum_k b(k)w(n-k)\right) x^*(n)\right] + E\left[\sum_k \sum_{\ell} b(k)b^*(\ell)w(n-k)w^*(n-\ell)\right] \\
&= K_X(0) - 2 \sum_k \operatorname{Re}\{b^*(k)K_{XW}(k)\} + \underbrace{\sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} b(k)b^*(\ell)K_W(\ell-k)}_{=\sigma_w^2 \sum_k |b(k)|^2} \\
&= K_X(0) + \sum_{k=-\infty}^{\infty} (|b(k)|^2 - 2\operatorname{Re}\{b^*(k)K_{XW}(k)\}) \\
&= K_X(0) + \underbrace{\sum_k |b(k) - K_{XW}(k)|^2}_{=0 \text{ if } b(k)=K_{XW}(k)} - \sum_k |K_{XW}(k)|^2 \\
&\quad w(n) = a(n) * y(n) \\
&\implies S_{XW}(f) = S_{XY}(f)A^*(f) \\
&\implies B(f) = S_{XY}(f)A^*(f)
\end{aligned}$$

$$y(n) \longrightarrow \underbrace{\mathbf{A}(f) \longrightarrow \mathbf{B}(f)}_{\mathbf{G}(f)} \longrightarrow \hat{x}(n)$$

$$\begin{aligned}
G(f) &= A(f)B(f) \\
&= A(f)S_{XY}(f)A^*(f) \\
&= S_{XY}(f)|A(f)|^2
\end{aligned}$$

Now, $y(n) \longrightarrow \mathbf{A}(f) \longrightarrow w(n)$

$$\begin{aligned}
S_W(f) &= S_Y(f)|A(f)|^2 \\
\implies |A(f)|^2 &= \frac{1}{S_Y(f)}
\end{aligned}$$

So,

$$\underbrace{G(f) = \frac{S_{XY}(f)}{S_Y(f)}}_{\text{Wiener Filter}}$$

Or,

$$\underbrace{G(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{H^*(f)S_X(f)}{H(f)S_X(f)H^*(f) + S_V(f)}}_{\text{Wiener Filter}}$$

Recall, LMMSE:

$$\text{zero mean } \underline{\mathbf{x}} \longrightarrow \mathbf{H} \longrightarrow \underbrace{\oplus}_{\uparrow \underline{\mathbf{v}}} \longrightarrow \underline{\mathbf{z}}, \quad \hat{\underline{\mathbf{x}}} = \mathbf{G}\underline{\mathbf{z}}$$

$$G_{matrix} = K_X H^{*T} [H K_X H^{*T} + K_V]^{-1}$$

(note similarity with $G(f)$)

$$x(n) \longrightarrow \mathbf{H}(\mathbf{f}) \longrightarrow \underbrace{\oplus}_{\uparrow v(n), E[v(n)]=0} \longrightarrow y(n) \longrightarrow \mathbf{G}(\mathbf{f}) \longrightarrow \hat{x}(n)$$

Suppose $x(n)$ is real,

$$\text{MSE} = E \left[(x(n) - \hat{x}(n))^2 \right] = E \left[\left(x(n) - \sum_k g(k)y(n-k) \right)^2 \right]$$

$$= E \left[x(n)^2 \right] - 2E \left[x(n) \sum_k g(k)y(n-k) \right] + E \left[\left(\sum_k g(k)y(n-k) \right)^2 \right]$$

$$\text{MSE} = K_X(0) - 2 \sum_k g(k)K_{XY}(k) + \sum_k \sum_\ell g(k)g(\ell)K_Y(k-\ell)$$

$$\frac{\partial}{\partial g(m)} = 0 = -2K_{XY}(m) + \underbrace{2 \sum_\ell g(\ell)K_Y(m-\ell)}_{\text{by symmetry of } K_Y \text{ for real sequences}}$$

So,

$$K_{XY}(m) = \sum_{\ell=-\infty}^{\infty} g(\ell)K_Y(m - \ell) \quad -\infty < m < \infty$$

$$\implies S_{XY}(f) = G(f)S_Y(f)$$

Or,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{H^*(f)S_X(f)}{H(f)S_X(f)H^*(f) + S_V(f)}$$

Mean Square Error

Orthogonal Principle \implies error is orthogonal to any function of the data.

$$E[\hat{x}^*(n)(x(m) - \hat{x}(m))] = 0$$

$$\implies E[\hat{x}(n)(x^*(m) - \hat{x}^*(m))] = 0 \quad \forall n, m$$

$$\text{MSE} = E[(x(n) - \hat{x}(n))(x^*(n) - \hat{x}^*(n))]$$

$$= E[|x(n)|^2] - E[x(n)\hat{x}^*(n)]$$

$$= K_X(0) - K_{X\hat{X}}(0)$$

In frequency domain,

$$K_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)df$$

$$K_{X\hat{X}}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{X\hat{X}}(f)df$$

$$K_X(k) \xleftrightarrow{F.T.} S_X(f)$$

$$K_{X\hat{X}}(k) \xleftrightarrow{F.T.} S_{X\hat{X}}(f)$$

$$K_{X\hat{X}}(k) = E[x(n)\hat{x}^*(n - k)]$$

$$= E[x(n) \sum_m g^*(m)y^*(n - k - m)]$$

$$= \sum_m g^*(m)K_{XY}(k + m)$$

$$\implies S_{X\hat{X}}(f) = G^*(f)S_{XY}(f)$$

$$= \left(\frac{S_{XY}(f)}{S_Y(f)} \right)^* S_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_Y(f)}$$

So,

$$\text{MSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[S_X(f) - \frac{|S_{XY}(f)|^2}{S_Y(f)} \right] df$$

Example

$$H(f) = \begin{cases} 1, & |f| \leq 0, \\ 0, & f_0 < |f| \leq \frac{1}{2} \end{cases}$$

$$S_X(f) = e^{-|f|}, \quad S_V(f) = \sigma_v^2$$

Determine optimum filter G (in mean square sense) to recover x given y .

Know,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)}$$

$$S_{XY}(f) = S_X(f)H^*(f)$$

So,

$$S_{XY}(f) = \begin{cases} e^{-|f|}, & |f| \leq 0, \\ 0, & f_0 < |f| \leq \frac{1}{2} \end{cases}$$

$$S_Y(f) = |H(f)|^2 S_X(f) + S_V(f)$$

$$S_Y(f) = \begin{cases} e^{-|f|} + \sigma_v^2, & |f| \leq 0, \\ \sigma_v^2, & f_0 < |f| \leq \frac{1}{2} \end{cases}$$

So,

$$G(f) = \begin{cases} \frac{e^{-|f|}}{e^{-|f|} + \sigma_v^2}, & |f| \leq 0, \\ 0, & f_0 < |f| \leq \frac{1}{2} \end{cases}$$

The Wiener Filter is non-causal!

Causal Constraint

$$g(n) = 0 \quad \forall n < 0$$

Get $\text{MSE} = K_X(0) - \sum_{k=0}^{\infty} K_{XW}^2(k)$.

18.2 Continuous Case

A similar analysis as above yields,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)}$$

$$\text{MSE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[S_X(\omega) - \frac{|S_{XY}(\omega)|^2}{S_Y(\omega)} \right] d\omega$$