17.0 Spectral Concepts

17.1 Spectral Densities

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-i2\pi fk} = \sum_{k=-\infty}^{\infty} K_X(k)e^{-i2\pi fk} + m_X^2\delta(f)$$

where $\delta(f)$ is a generalized function defined by its sifting property

$$\int_{-\infty}^{\infty} x(f)\delta(f - f_0)df = x(f_0)$$

provided $x(f)$ is a function continuous at $f = f_0$. So

$$R_X(n) = K_X(n) + \int_{-1/2}^{1/2} m_X^2\delta(f)e^{i2\pi fn}df$$

$$= K_X(n) + m_X^2.$$

In general, if

$$R_X(n) = K_X(n) + \sum_k a_k e^{-i2\pi f_k}$$

then

$$S_X(f) = \tilde{S}_X(f) + \sum_k a_k\delta(f - f_k)$$

where

$$K_X(n) \longleftrightarrow \tilde{S}_X(f).$$

17.2 Spectral Factorization

System Function

Consider

$$H(z) = \sum_n h(n)z^{-n}.$$

The region of convergence (ROC) of this z-transform is

$$ROC = \left\{z : \sum_n |h(n)z^{-n}| < \infty\right\}.$$

This guarantees uniform convergence.

Causal Sequences

Here $h(n) = 0$ for $n < 0$. Then

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots .$$

Note that $h(n)$ is causal if and only if $H(z)$ converges as $|z| \to \infty$.

Stable Causal Sequences

$h(n)$ is causal and stable if

$$\sum_{n=0}^{\infty} |h(n)| < \infty.$$

This is equivalent to

$$\sum_{n=0}^{\infty} |h(n)z^{-n}|_{|z|=1} < \infty.$$

Thus causal $h(n)$ is stable if and only if $H(z)$ converges on the unit circle in the z-plane.

Poles

Here we identify those values of z that make $H(z) \to \infty$.

Example:

$$h(n) = \alpha^n u(n),\ |\alpha| < 1.$$

$$H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1 - \alpha z^{-1}}.$$

To find the pole we set $(1 - \alpha z^{-1}) = 0$ to get $\alpha = z$.

Zeros

Here we identify those values of z that make $H(z) = 0$.

Example:

$$h(n) = \delta(n) - \alpha\delta(n-1).$$

$$H(z) = (1 - \alpha z^{-1}.)$$

To find the zero we set $(1 - \alpha z^{-1}) = 0$ to get $\alpha = z$.

Poles and ROC

ROC for causal sequences= $\{z : d_{max} < |z| < \infty\}$ where d_{max} is the magnitude of the largest pole.

Poles and Stability

Since ROC must include the unit circle all poles must lie inside the unit circle.

Rational System Function

Consider the linear difference equation

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k).$$

Take z-transform

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) - \sum_{k=1}^{N} a_k z^{-k} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}}.$$

Poles occur at roots of

$$\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = d_j,\ \ j = 1, 2, \ldots, N.$$

Zeros occur at roots of

$$\left(\sum_{k=0}^{M} b_k z^{-k}\right) = c_j,\ \ j = 1, 2, \ldots, M.$$

$$S_Y(f) = |H(f)|^2 S_X(f) = H(f)H^*(f)S_X(f)$$
$$= H(z)H^*(z)S_X(f)\Big|_{z=e^{i2\pi f}}.$$

If $h(n)$ is real then $H(z) = H^*(z^*) \Rightarrow H^*(z) = H(z^*)$. Now

$$H^*(z)\Big|_{z=e^{i2\pi f}} = H^*\left(e^{i2\pi f}\right) = H\left(e^{-i2\pi f}\right) = H(z^{-1})\Big|_{z=e^{i2\pi f}}.$$

So for $h(n)$ real

$$S_Y(f) = H(z)H(z^{-1})S_X(f)\Big|_{z=e^{i2\pi f}}.$$

Example: Let $x(n)$ be an i.i.d. sequence with mean zero and variance $\sigma^2 = 1$. $x(n)$ is applied to a filter with z-transform $H(z)$. The output is $y(n)$. Say

$$K_Y(m,n) = \alpha^{|m-n|},\ |\alpha| < 1 \text{ and is real.}$$

Find $H(z)$.

Let $k = m - n$. Then

$$K_Y(k) = \alpha^{|k|}.$$

$$S_Y(f) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-i2\pi fk}$$

$$= \sum_{k=0}^{\infty} \alpha^k e^{-i2\pi fk} + \sum_{k=-\infty}^{0} \alpha^{-k} e^{-i2\pi fk} - 1$$

$$= \sum_{k=0}^{\infty} \left(\alpha e^{-i2\pi f}\right)^k + \sum_{k=0}^{\infty} \left(\alpha e^{i2\pi f}\right)^k - 1$$

$$= \frac{1}{1-\alpha e^{-i2\pi f}} + \frac{1}{1-\alpha e^{i2\pi f}} - 1$$

$$= \frac{1-\alpha^2}{\left(1-\alpha e^{-i2\pi f}\right)\left(1-\alpha e^{i2\pi f}\right)}.$$

Replace $e^{i2\pi f}$ by z. Then

$$S_Y(f) = \frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}.$$

We have poles at $z = \alpha$ and $z = 1/\alpha$. We can write

$$S_Y(z) = H(z)H(z^{-1})$$

where

$$H(z) = \frac{(1-\alpha^2)^{1/2}}{1-\alpha z^{-1}}.$$