16.0 Stochastic Inputs to LSI Systems

16.1 General Systems

$$y(u,n) = \sum_{m=0}^{\infty} h(m,n)x(u,m)$$

$h(m,n)$ is not shift-invariant in general and $y(u,n)$ does not exist in general (there may be some sample processes that are not summable).

$y(u,n)$ exists in the mean square sense if

1) BIBO
2) Variances are bounded

Let $y_k(u,n) = \sum_{m=0}^{k} h(m,n)x(u,m)$. If for every $\epsilon > 0 \quad \exists$ a number $N(\epsilon)$ $s.t.$

$$E[|y_k(u,n) - y_\ell(u,n)|^2] < \epsilon \quad \forall k, \ell > N(\epsilon)$$

then

$$y_k(u,n) \xrightarrow{m.s.} y(u,n)$$

Consider WLOG that $k > \ell$,

$$E\left[|y_k(n) - y_\ell(n)|^2\right] = E\left[\left|\sum_{m=\ell+1}^{k} h(m,n)x(m)\right|^2\right]$$

$$= \sum_{m=\ell+1}^{k} \sum_{p=\ell+1}^{k} h(m,n)h^*(n,p)R_X(n,p)$$

$$\leq \sum_{m=\ell+1}^{k} \sum_{p=\ell+1}^{k} |h(m,n)||h^*(n,p)||R_X(n,p)|$$

$$\leq [\sum_{p=\ell+1}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}}]^2$$

If

$$\lim_{k\to\infty} \sum_{p=0}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}}$$

exists, then as $k, \ell \longrightarrow \infty$

$$\sum_{p=\ell+1}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}} \longrightarrow 0$$

Theorem: If $X(u,n)$ is a sequence of random variables with $R_X(n,n) < \infty \quad \forall n$ and $h(n,m)$ is absolutely summable, i.e.,

$$\sum_{m=0}^{\infty} |h(m,n)| < \infty \quad \forall n$$

then,

$$\sum_{m=0}^{\infty} |h(n,m)| x(u,m)$$

exists in the mean square sense.

16.2 WSS in LSI Systems

Assume $h(n)$ is causal, LSI, BIBO stable. Recall we have BIBO stability and causality *if and only if* $\sum_{n=0}^{\infty} |h(n)| < \infty$, and $h(n) = 0$ for $n < 0$. Assume $x(n) is\ WSS$ and $E[x(n)^2] < \infty$.

Mean

$$E[y(n)] = E[h(n) * x(n)]$$

$$= \sum_{k=0}^{\infty} h(k) E[x(n-k)]$$

$$\mu_y = \mu_x \sum_{k=0}^{\infty} h(k) \quad \textit{(constant)}$$

Cross Correlation

$$R_{XY}(n,m) = E[x(u) y^*(m)]$$

$$= E[x(u) \sum_{k=0}^{\infty} h^*(k) x^*(m-k)]$$

$$= \sum_{k=0}^{\infty} h^*(k) R_X(n, m-k)$$

$$= \sum_{k} h^*(k) R_X(n-m+k)$$

Let $\ell = n - m$

$$R_{XY}(\ell) = \sum_{k=0}^{\infty} h^*(k) R_X(\ell + k)$$
$$= R_X(\ell) * h^*(-\ell)$$

Similar derivation for cross covariance, $K_{XY}(n, m)$

<u>Correlation of Y</u>

$$R_Y(n, m) = E[y(n)y^*(m)]$$
$$= E[\sum_p h(p)x(n-p)y^*(m)]$$
$$= \sum_{p=0}^{\infty} h(p) \underbrace{E[x(n-p)y^*(m)]}_{R_{XY}(n-p-m)}$$
$$= \sum_{p=0}^{\infty} h(p) \sum_{k=0}^{\infty} h^*(k) R_X(n-p-m+k)$$

Let $q = n - m$

$$R_Y(q) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} h(p)h^*(k)R_X(q-p+k)$$
$$= h(q) * R_X(q) * h^*(-q)$$

Recall

$R_Y = HR_XH^\dagger$ *(note similarity between time domain and freq. domain).*

Similar derivation for covariance of $Y, K_Y(n, m)$.

<u>Fourier Space</u>

Define a cross-spectral density as

$$S_{XY}(f) = \sum_{n=-\infty}^{\infty} R_{XY}(n) e^{-i2\pi f n}, f \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} h^*(k) R_X(n+k) e^{-i2\pi f(n+k)} e^{i2\pi f k}$$

or

$$S_{XY}(f) = S_X(f)H^*(f).$$

Now,

$$S_Y(f) = \sum_{n=-\infty}^{\infty} R_Y(n)e^{-i2\pi fn}$$
$$= H(f)H^*(f)S_X(f).$$

$$S_Y(f) = |H(f)|^2 S_X(f).$$

Recall, $R_Y(q) = h(q) * h^*(-q) * R_X(q)$

Ex1 Suppose $x(n)$ is an i.i.d., zero-mean sequence, var $= \sigma_X^2$.

$$K_X(n) = \sigma_X^2 \delta(n)$$

$$S_X(f) = \sum_{n=-\infty}^{\infty} K_X(n)e^{-i2\pi fn}$$
$$= \sigma_X^2 \quad \longrightarrow \quad \textbf{"White Noise"}$$

(power equally spread over all frequences)

Consider the moving average

$$y(n) = \sum_{k=0}^{M} b_k x(n-k).$$

Then

$$H(f) = \sum_{k=0}^{M} b_k e^{-i2\pi fk}.$$

Thus

$$S_Y(f) = \left| \sum_{k=0}^{M} b_k e^{-i2\pi fk} \right|^2 \sigma_X^2.$$

Now consider the special case $M = 1,\ b_0 = h(0) = 1,\ b_1 = h(1) = -1$ and $h(n) = 0$ for $n \neq 0, 1$. Then

$$y(n) = x(n) - x(n-1).$$

So

$$H(f) = \sum_{k=0}^{1} h(k) e^{-i2\pi fk} = 1 - e^{-i2\pi f}.$$

$$|H(f)|^2 = 2 - 2\cos(2\pi f)$$

and

$$S_Y(f) = \sigma_X^2 (2 - 2\cos(2\pi f)).$$