15.0 Wide Sense Stationary (WSS) Random Processes

15.1 Definitions

Definition: A random process $X(u,t)$ is stationary if

$$F(\mathcal{T}_\tau(\underline{x})) = F(\underline{x})$$

Definition: A random process is wide sense stationary if it has time invariant 1^{st} and 2^{nd} order statistics.

So, 1) $\mu_X(t) = \mu_X(t+\tau)$ Let $\tau = -t$

$$\mu_X(t) = \mu_X(0) \longrightarrow \quad \textit{some constant}$$

2) $R_X(t_1,t_2) = R_X(t_1+\tau, t_2+\tau)$ Let $\tau = -t_2$

$$R_X(t_1,t_2) = R_X(t_1-t_2, 0) = R_X(t_1-t_2)$$

15.2 Power Spectral Density (PSD) in Discrete Time Systems

Positive Semi-Definite Property

$$\sum_{k=-N}^{N} \sum_{\ell=-N}^{N} a_k R_X(k,\ell) a_\ell^* \geq 0 \quad \forall N \in \mathcal{T} \quad \textit{(discrete)}$$

Let $a_k = e^{-i2\pi fk}$. Assume $X(u,n)$ is WSS.

$$\sum_{k=-N}^{N} \sum_{\ell=-N}^{N} e^{-i2\pi f(k-\ell)} R_X(k-\ell) \geq 0$$

Let $m = k-\ell$; $n = k+\ell$

$$\Longrightarrow \sum_{m=-2N}^{2N} (2N+1-|m|) e^{-i2\pi fm} R_X(m) \geq 0 \quad \textit{(think)}$$

divide by $(2N+1)$, take limit,

$$\lim_{N\to\infty} \sum_{m=-2N}^{2N} (1 - \frac{|m|}{2N+1}) e^{-i2\pi fm} R_X(m) \geq 0$$

Define

$$\underbrace{S_X(f) = \sum_{m=-\infty}^{\infty} R_X(m)e^{-i2\pi fm},}_{\textbf{Power Spectral Density}} \qquad f \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

So, PSD = Fourier Transform of autocorrelation function.

Properties

1) $R_X(m)$ are Fourier Series coefficient for $S_X(f)$.

$$R_X(m) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)e^{i2\pi fm}df$$

$$R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)df$$

$$R_X(0) = E[|X(u)|^2]$$

2) Periodic: $S_X(f) = S_X(f+k)$ for integer k.

3) Any P.S.D. correlation has a non-negative PSD.

4) $S_X(f)$ is real.

5) If $R_X(u)$ is real, then $S_X(f) = S_X(-f)$.