8.0 Projections and K-L Expansions

8.1 Projection of Random Vectors

Definition: A vector $\mathbf{e} = (\mathbf{e_1}, \mathbf{e_n})^t$ is called a *directional vector* if $\|\mathbf{e}\| = 1$. Here

$$\|\mathbf{e}\|^2 = \sum_{i=1}^{n} |\mathbf{e}_i|^2$$

Definition: The projection of a random vector $\mathbf{X}(u)$ in a direction $\mathbf{e}$ is

$$\mathbf{X}_e\,(u) = <\mathbf{X}(u), \mathbf{e}> = \mathbf{e}^\dagger\,\mathbf{X}(u)$$

If $\mu_X = 0$, get

$$\sigma^2_{X_e} = \mathbf{e}^\dagger \mathbf{K}_X \mathbf{e}$$

Suppose $\mathbf{e}$ is a normalized eigenvector of $\mathbf{K}_X$ with eigenvalue λ, then

$$\sigma^2_{X_e} = \lambda \|\mathbf{e}\|^2 = \lambda$$

Now let $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$ be an orthonormal basis of an n-dimensional vector space where each $\mathbf{e}_k$ is an eigenvector of $\mathbf{K}_X$ with eigenvalue λ_k. Then, with $\mathbf{e}$ as a directional vector

$$\mathbf{e} = \sum_{k=1}^{n} a_k \mathbf{e}_k, \quad \textit{some } a_k$$

or, $\quad \mathbf{e} = \mathbf{E}\mathbf{a} \quad \Longrightarrow \quad \mathbf{a} = \mathbf{E}^{-1}\mathbf{e}\ , \quad$ where $\mathbf{a} = (\mathbf{a}_1, \cdots \mathbf{a}_n)^t$
or,

$$\mathbf{a}_k = \mathbf{e}_k^\dagger\,\mathbf{e}$$

$\mathbf{a}_k$ is the projection of $\mathbf{e}$ onto the k^{th} eigenvector.

We have

$$1 = \|\mathbf{e}\|^2 = \|\mathbf{E}\mathbf{a}\|^2 = (\mathbf{E}\mathbf{a})^\dagger \mathbf{E}\mathbf{a} = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{E}\mathbf{a} = \mathbf{a}^\dagger \mathbf{a} = \|\mathbf{a}\|^2$$

So the transformation is norm-preserving.

With $\quad \mathbf{e} = \sum_{k=1}^{n} \mathbf{a}_k \mathbf{e}_k, \quad$ we get

$$\sigma^2_{X_e} = \mathbf{e}^\dagger \mathbf{K}_X \mathbf{e}$$

$$= (\sum_j \mathbf{a}_j \mathbf{e}_j)^\dagger \mathbf{K}_X (\sum_k \mathbf{a}_k \mathbf{e}_k)$$

$$= \sum_j \sum_k \mathbf{a}_j^* \mathbf{a}_k \mathbf{e}_j^\dagger \mathbf{K}_X \mathbf{e}_k$$

$$= \sum_j \sum_k \mathbf{a}_j^* \mathbf{a}_k \mathbf{e}_j^\dagger \lambda_k \mathbf{e}_k$$

$$= \sum_j \sum_k \lambda_k \mathbf{a}_j^* \mathbf{a}_k < \mathbf{e}_k, \mathbf{e}_j >$$

$$= \sum_j \sum_k \lambda_k \mathbf{a}_j^* \mathbf{a}_k \delta(k-j)$$

$$= \sum_k \lambda_k |\mathbf{a}_k|^2$$

But, $\sum_k |\mathbf{a}_k|^2 = 1$, so (think why?)

$$\lambda_{min} \leq \sigma_{X_e}^2 \leq \lambda_{max}$$

or, $\mathbf{e} = \mathbf{E}\mathbf{a} \Longrightarrow \mathbf{e}^\dagger \mathbf{X} = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{X}$

$$\sigma_{X_e}^2 = E[\mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{X} \mathbf{X}^\dagger \mathbf{E} \mathbf{a}] = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{K}_X \mathbf{E} \mathbf{a}$$

$$= \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{E} \Lambda \mathbf{a} = \mathbf{a}^\dagger \Lambda \mathbf{a} = \sum_k \lambda_k |\mathbf{a}_k|^2$$

$$\Longrightarrow \lambda_{min} \leq \sigma_{X_e}^2 \leq \lambda_{max}$$

8.2 Karhunen-Loeve Expansion

$$E[|\mathbf{X}|^k] = \int |x|^k f_X(x)dx \geq \int_{|x|\geq\alpha} |x|^k f_X(x)dx$$
$$\geq \alpha^k \int_{|x|\geq\alpha} f_X(x)dx \quad (\alpha > 0)$$
$$= \alpha^k P(|X| \geq \alpha)$$

So, $P(|X| \geq \alpha) \leq \frac{1}{\alpha^k} E[|X|^k]$. This is Markov's Inequality.

If we set k=2 and subtract $\mu = E(X)$ from X, we get Chebyshev's Inequality:

$$P(|X - \mu| > \alpha) \leq \frac{1}{\alpha^2} \underbrace{E(|X-\mu|^2)}_{\text{Var}(X)}$$

Theorem: Let $\mathbf{X}(u)$ be a zero-mean random vector and let $\widetilde{\mathbf{X}}_k(u) =< \mathbf{X}(u), \mathbf{e}_k >$ where the $\mathbf{e}_k$ is an orthonormal basis of eigenvectors of $\mathbf{K}_X$. Then,

$$\mathbf{X}(u) = \sum_{k=1}^{n} \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k \qquad \textit{(K-L expansion in probability)}$$

Furthermore,

$$E[\widetilde{\mathbf{X}}_k(u)] = 0$$

and,

$$E[\widetilde{\mathbf{X}}_k(u)\widetilde{\mathbf{X}}^*_m(u)] = \lambda_m \delta(m-k)$$

Proof: Clearly $E[\widetilde{\mathbf{X}}_k(u)] = 0$.

$$E[\widetilde{\mathbf{X}}_k(u)\widetilde{\mathbf{X}}^*_m(u)] = E[< \mathbf{X}(u), \mathbf{e}_k >< \mathbf{X}(u), \mathbf{e}_m >^*]$$
$$= E[(\mathbf{e}_k^\dagger \mathbf{X}(u))(\mathbf{e}_m^\dagger \mathbf{X}(u))^*]$$
$$= E[\mathbf{e}_k^\dagger \mathbf{X}(u)\mathbf{X}^\dagger(u)\mathbf{e}_m]$$
$$= \mathbf{e}_k^\dagger \mathbf{K}_x \mathbf{e}_m$$
$$= \mathbf{e}_k^\dagger \lambda_m \mathbf{e}_m$$

$$= \lambda_m \delta(m-k)$$

To say $\mathbf{X}(u) = \sum_{k=1}^{n} \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k$ in probability we mean for any $\epsilon > 0$

$$P[\,\|\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\| \geq \epsilon] = 0 \qquad \textit{(for any } \epsilon > 0\textit{)}$$

Now, by Chebyshev's Inequality,

$$P[\,\|\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\| \geq \epsilon] \leq \frac{E\left[||\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k||\right]^2}{\epsilon^2}$$

$$E\|\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\|^2 = \mathbf{E}\left[\left(\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\right)^\dagger \left(\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\right)\right]$$

$$= E\left[\|\mathbf{X}(u)\|^2 - \mathbf{X}^\dagger(u)\sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k - \left(\sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\right)^\dagger \mathbf{X}(u)\right.$$

$$\left. + \left(\sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\right)^\dagger \left(\sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\right)\right]$$

Since

$$\widetilde{\mathbf{X}}_k(u) = \mathbf{e}_k^\dagger \mathbf{X}(u)$$

We have that

$$\mathbf{X}^\dagger(u)\sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k = \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{X}^\dagger(u)\mathbf{e}_k$$

$$= \sum_k \widetilde{\mathbf{X}}_k(\mathbf{e}_k^\dagger \mathbf{X}(u))^\dagger = \sum_k \widetilde{\mathbf{X}}_k \widetilde{\mathbf{X}}_k^*$$

$$= \sum_k |\widetilde{\mathbf{X}}_k|^2$$

Therefore,

$$E\|\mathbf{X}(u) - \sum_k \widetilde{\mathbf{X}}_k(u)\mathbf{e}_k\|^2 = E[\|\mathbf{X}(u)\|^2] - 2\sum_k E|\tilde{\mathbf{X}}_k(u)|^2 + \sum_k \sum_m E[\widetilde{\mathbf{X}}_k^*(u)\widetilde{\mathbf{X}}_m^(u)]\delta(m-k)$$

$$= \mathsf{Tr}(\mathbf{K}_X) - 2\sum_k E|\tilde{\mathbf{X}}_k(u)|^2 + \sum_k E|\widetilde{\mathbf{X}}_k(u)|^2$$

$$
\begin{aligned}
&= \mathsf{Tr}(\mathbf{K}_X) - \sum_k E|\tilde{\mathbf{X}}_k(u)|^2 \\
&= \sum_{k=1}^{n} \lambda_k - \sum_{k=1}^{n} \lambda_k = 0
\end{aligned}
$$