5.0 Spectral Shaping Problem

Consider
$$\mathbf{X} = \mathbf{HW} + \mathbf{a}$$
where $\mathbf{W}$ is mean 0 and $\mathbf{K_W} = \mathbf{I}$. We know
$$\mu_\mathbf{X} = \mathbf{a}, \quad \mathbf{K_X} = \mathbf{HK_WH}^\dagger = \mathbf{HH}^\dagger.$$
Finding $\mathbf{H}$ to give us some desired $\mathbf{K_X}$ is called *spectral shaping*.

The diagonal form of $\mathbf{K_X}$ is
$$\mathbf{K_X} = \mathbf{E_X}\Lambda_\mathbf{X}\mathbf{E}_\mathbf{X}^\dagger.$$
Let
$$\Lambda_\mathbf{X}^{1/2} = diag\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\right), \; \lambda_i \geq 0 \; real.$$
Then,
$$\mathbf{K_X} = \mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\Lambda_\mathbf{X}^{1/2}\mathbf{E}_\mathbf{X}^\dagger = \left(\mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\right)\left(\mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\right)^\dagger.$$
Let
$$\mathbf{H} = \mathbf{E_X}\Lambda_\mathbf{X}^{1/2}$$
then
$$\mathbf{K_X} = \mathbf{HH}^\dagger$$
as desired. So this $\mathbf{H}$ is a solution to the spectral shaping problem. This solution is not unique. Consider
$$\mathbf{H} = \mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\mathbf{U}$$
where $\mathbf{U}$ is any unitary matrix. Then,
$$\mathbf{HH}^\dagger = \left(\mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\mathbf{U}\right)\left(\mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\mathbf{U}\right)^\dagger$$
$$= \mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\mathbf{UU}^\dagger\Lambda_\mathbf{X}^{1/2}\mathbf{E}_\mathbf{X}^\dagger$$
$$= \mathbf{E_X}\Lambda_\mathbf{X}^{1/2}\mathbf{I}\Lambda_\mathbf{X}^{1/2}\mathbf{E}_\mathbf{X}^\dagger$$
$$= \mathbf{E_X}\Lambda_\mathbf{X}\mathbf{E}_\mathbf{X}^\dagger$$
$$= \mathbf{K_X}$$

so, $\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}$ is also a solution.

<u>**Claim:**</u> $\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}$ gives us all solutions to our spectral shaping problem, where $\mathbf{U}$ is a unitary matrix.

Proof: Let $\tilde{\mathbf{H}}$ be a solution to the spectral shaping problem. Then

$$\mathbf{K_X} = \tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\dagger}.$$

Let

$$\tilde{\mathbf{U}} = \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}$$

so

$$\tilde{\mathbf{H}} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\tilde{\mathbf{U}}.$$

Then,

$$\begin{aligned}
\tilde{\mathbf{U}}\tilde{\mathbf{U}}^{\dagger} &= \left(\Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\right)\left(\Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\right)^{\dagger} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}\mathbf{E}_{\mathbf{X}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{I}\Lambda_{\mathbf{X}}\mathbf{I}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\Lambda_{\mathbf{X}}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \mathbf{I}.
\end{aligned}$$

So, $\tilde{\mathbf{U}}$ is a unitary matrix, call it $\mathbf{U}$. Hence,

$$\tilde{\mathbf{H}} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}.$$