

4.0 Analysis of Correlation Matrices

4.1 Decomposition of the Correlation Matrix

Recall \mathbf{R}_X is non-negative definite.

Proposition: If a matrix \mathbf{M} is Hermitian symmetric then

- i. all its eigenvalues are real.
- ii. the eigenvectors corresponding to distinct eigenvalues are orthogonal.
- iii. there exists an orthonormal basis of the n-dimensional vector space which consists of the normalized eigenvectors of \mathbf{M} .
- iv. \mathbf{M} is non-negative definite if and only if all its eigenvalues are non-negative.

Proof:

- i. Let λ be an eigenvalue and \mathbf{e} a normalized eigenvector such that $\|\mathbf{e}\| = 1$.
Then

$$\mathbf{e}^\dagger \mathbf{M} \mathbf{e} = \mathbf{e}^\dagger \lambda \mathbf{e} = \lambda \|\mathbf{e}\|^2.$$

$$\mathbf{e}^\dagger \mathbf{M} \mathbf{e} = (\mathbf{e}^\dagger \mathbf{M}) \mathbf{e} = (\mathbf{M}^\dagger \mathbf{e})^\dagger \mathbf{e} = (\mathbf{M} \mathbf{e})^\dagger \mathbf{e} = (\lambda \mathbf{e})^\dagger \mathbf{e} = \lambda^* \|\mathbf{e}\|^2.$$

Thus, $\lambda = \lambda^*$ so λ is real.

- ii Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of \mathbf{M} and let $\mathbf{e}_1, \mathbf{e}_2$ be the corresponding eigenvectors. Then,

$$\mathbf{e}_1^\dagger \mathbf{M} \mathbf{e}_2 = \mathbf{e}_1^\dagger \lambda_2 \mathbf{e}_2 = \lambda_2 \langle \mathbf{e}_1, \mathbf{e}_2 \rangle .$$

$$\mathbf{e}_1^\dagger \mathbf{M} \mathbf{e}_2 = (\mathbf{e}_1^\dagger \mathbf{M}) \mathbf{e}_2 = (\mathbf{M} \mathbf{e}_1)^\dagger \mathbf{e}_2 = \lambda_1^* \mathbf{e}_1^\dagger \mathbf{e}_2 = \lambda_1 \langle \mathbf{e}_1, \mathbf{e}_2 \rangle .$$

Thus

$$(\lambda_2 - \lambda_1) \langle \mathbf{e}_1^\dagger, \mathbf{e}_2 \rangle = 0 \Rightarrow \langle \mathbf{e}_1^\dagger, \mathbf{e}_2 \rangle = 0.$$

- iii. This follows from ii. when all eigenvalues are real. For the general case see a text on linear algebra.

iv. Show

$$\mathbf{x}^\dagger \mathbf{M} \mathbf{e} \geq 0 \Leftrightarrow \text{all } \lambda_i \geq 0.$$

“ \Rightarrow ” only if part or necessity. Assume $\mathbf{x}^\dagger \mathbf{M} \mathbf{e} \geq 0 \quad \forall$ vectors \mathbf{x} . Let $\mathbf{x} = \mathbf{e}$. Then

$$\mathbf{M} \mathbf{e} = \lambda \mathbf{e} \Rightarrow \mathbf{e}^\dagger \mathbf{M} \mathbf{e} = \lambda \|\mathbf{e}\|^2.$$

$$\mathbf{e}^\dagger \mathbf{M} \mathbf{e} \geq 0 \Rightarrow \lambda \|\mathbf{e}\|^2 \geq 0 \Rightarrow \lambda \geq 0.$$

“ \Leftarrow ” if part or sufficiency. See a text on linear algebra.

4.2 The Spectral Decomposition of the Correlation Matrix

We have a random vector $\mathbf{X}(u)$ with corresponding correlation matrix $\mathbf{R}_\mathbf{X}$. We know $\mathbf{R}_\mathbf{X}$ is non-negative definite. So there exists an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of eigenvectors of $\mathbf{R}_\mathbf{X}$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Then

$$\mathbf{R}_\mathbf{X} \mathbf{e}_i = \lambda_i \mathbf{e}_i, \quad i = 1, 2, \dots, n.$$

Then

$$\mathbf{R}_\mathbf{X} [\mathbf{e}_1 \dots \mathbf{e}_n] = [\lambda_1 \mathbf{e}_1 \dots \lambda_n \mathbf{e}_n].$$

Now let

$$\mathbf{E} = [\mathbf{e}_1 \dots \mathbf{e}_n], \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}.$$

We can then write

$$\mathbf{R}_\mathbf{X} \mathbf{E} = \mathbf{E} \Lambda \Rightarrow \mathbf{R}_\mathbf{X} = \mathbf{E} \Lambda \mathbf{E}^{-1}.$$

Note that \mathbf{E} is a unitary matrix.

Definition: A square matrix \mathbf{U} is said to be a *unitary matrix* if

$$\mathbf{U} \mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}.$$

So $\mathbf{U}^{-1} = \mathbf{U}^\dagger$.

Claim: For \mathbf{U} unitary, every eigenvalue of \mathbf{U} is such that $|\lambda| = 1$.

Proof: Let \mathbf{x} be an eigenvector of \mathbf{U} . Then

$$\mathbf{U} \mathbf{x} = \lambda \mathbf{x}.$$

But

$$(\mathbf{U}\mathbf{x})^\dagger (\mathbf{U}\mathbf{x}) = \mathbf{x}^\dagger \mathbf{U}^\dagger \mathbf{U} \mathbf{x} = \mathbf{x}^\dagger \mathbf{x} \Rightarrow \|\mathbf{U}\mathbf{x}\|^2 = \|\mathbf{x}\|^2 \Rightarrow \|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|.$$

So

$$\|\mathbf{U}\mathbf{x}\| = \|\lambda\mathbf{x}\| \Rightarrow \|\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\| \Rightarrow |\lambda| = 1.$$

By computing $\mathbf{E}^\dagger \mathbf{E}$ we find that \mathbf{E} is unitary so that $\mathbf{E}^{-1} = \mathbf{E}^\dagger$. Hence

$$\mathbf{R}_\mathbf{X} = \mathbf{E} \Lambda \mathbf{E}^\dagger.$$

The above is called the *Spectral Decomposition of $\mathbf{R}_\mathbf{X}$* . We may write the last expression in the form

$$\mathbf{R}_\mathbf{X} = \sum_{i=1}^n \lambda_i \mathbf{e}_i \mathbf{e}_i^\dagger.$$

The above is called the *Spectral Expansion of $\mathbf{R}_\mathbf{X}$* .