

3.0 Gaussian Random Vectors

An important class of random vectors is the Gaussian (or normal) random vectors.

3.1 Gaussian Random Variables

Definition: A random variable X is a *Gaussian random variable* if its density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in (-\infty, \infty).$$

We write $X \sim N(\mu, \sigma^2)$.

The distribution function for X is

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-\mu)^2/2\sigma^2} dt.$$

Results from the Theory of Equations prove there can be no closed form expression for this integral, i.e., there does not exist a function containing a finite number of terms whose derivative equals $f_X(x)$. So to evaluate this integral we have to make use of tables or series approximations or functional approximations.

Definition: The *characteristic function* of a random variable X is given by

$$\Phi_X(\omega) = E[e^{i\omega X}] = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) dx, \quad i = \sqrt{-1}.$$

This is a Fourier transform (with a sign change).

If $X \sim N(\mu, \sigma^2)$, then

$$\Phi_X(\omega) = \exp\left(i\omega\mu - \frac{\sigma^2\omega^2}{2}\right).$$

Since Fourier transform pairs are unique we get

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-i\omega x} d\omega.$$

Definition: The *moment generating function* of a random variable X is given by

$$M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx.$$

We may thus write

$$M_X(s) = \sum_{n=0}^{\infty} \frac{\mu_X^{(n)}}{n!} s^n, \quad \text{where } \mu_X^{(n)} = E[X^n]$$

provided all the moments are finite. We get

$$\mu_X^{(n)} = \left. \frac{d^n M_X(s)}{ds^n} \right|_{s=0}.$$

For $X \sim N(\mu, \sigma^2)$,

$$M_X(s) = \exp \left[\mu s + \frac{\sigma^2 s^2}{2} \right]$$

$$\left. \frac{dM_X(s)}{ds} \right|_{s=0} = \left. (\mu + \sigma^2 s) e^{\mu s + \sigma^2 s^2 / 2} \right|_{s=0} \Rightarrow \mu_X = \mu.$$

Also,

$$\begin{aligned} \left. \frac{d^2 M_X(s)}{ds^2} \right|_{s=0} &= \left. (\mu + \sigma^2 s)^2 e^{\mu s + \sigma^2 s^2 / 2} + \sigma^2 e^{\mu s + \sigma^2 s^2 / 2} \right|_{s=0} \\ &\Rightarrow \mu_X^{(2)} = \mu^2 + \sigma^2. \end{aligned}$$

Thus,

$$\text{Var}(X) = \sigma^2.$$

3.2 Gaussian Random Vector

Definition: A random vector

$$\mathbf{W} = [W_1(u) \cdots W_n(u)]^t$$

is said to be a *white Gaussian random vector* if the random variables $W_1(u), \dots, W_n(u)$ are independent and identically distributed (i.i.d.) Gaussian random variables with distribution $W_i \sim N(\mu, \sigma^2)$.

We will often make $\mu = 0$ and $\sigma^2 = 1$ since the data can always be translated and scaled so that this is so.

Thus,

$$\begin{aligned}\mu_W &= (0, \dots, 0)^t. \\ \mathbf{R}_W &= \mathbf{K}_W = \sigma^2 \mathbf{I}_n. \\ f_W(x_1, \dots, x_n) &= \prod_{i=1}^n f_{W_i}(x_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_i^2}{2\sigma^2}\right) \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \prod_{i=1}^n \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{\|\mathbf{x}\|^2}{2\sigma^2}\right)\end{aligned}$$

where,

$$\mathbf{x}^t = (x_1, \dots, x_n), \quad \|\mathbf{x}\|^2 = \mathbf{x}^t \mathbf{x}.$$

Definition: A *Gaussian random vector* $\mathbf{X}(u)$ is defined as an affine transformation of the white Gaussian random vector.

We have

$$\begin{aligned}\mathbf{X}(u) &= \mathbf{H}\mathbf{W}(u) + \mathbf{a}. \\ \mu_{\mathbf{X}} &= E[\mathbf{H}\mathbf{W}(u) + \mathbf{a}] = \mathbf{a}. \\ \mathbf{K}_{\mathbf{X}} &= \mathbf{H}\mathbf{K}_W\mathbf{H}^\dagger = \mathbf{H}\sigma^2\mathbf{I}_n\mathbf{H}^\dagger = \sigma^2\mathbf{H}\mathbf{H}^\dagger.\end{aligned}$$

If $\sigma^2 = 1$ then

$$\mathbf{K}_{\mathbf{X}} = \mathbf{H}\mathbf{H}^\dagger.$$

3.3 Characteristic Function of a Gaussian Random Vector

Let

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{a}, \quad \mathbf{K}_W = \mathbf{I}_n.$$

Then

$$\begin{aligned}\Phi_X(\omega) &= E\left[\exp\left\{i\omega^t \mathbf{X}\right\}\right], \quad \omega = (\omega_1, \dots, \omega_n)^t \\ &= E\left[\exp\left\{i\omega^t (\mathbf{H}\mathbf{W}(u) + \mathbf{a})\right\}\right] = \exp\left\{i\omega^t \mathbf{a}\right\} \Phi_W(\mathbf{H}^t \omega).\end{aligned}$$

Now

$$\begin{aligned}\Phi_W(\omega) &= E \left[\exp \left\{ i\omega^t \mathbf{W}(u) \right\} \right] = E \left[\exp \left\{ i \sum_{k=1}^n \omega_k W_k(u) \right\} \right] \\ &= \prod_{k=1}^n \Phi_{W_k}(\omega) = \exp \left\{ -\frac{1}{2} \sigma^2 \|\omega\|^2 \right\} = \exp \left\{ -\frac{1}{2} \|\omega\|^2 \right\}.\end{aligned}$$

Thus,

$$\Phi_X(\omega) = \exp \left\{ i\omega^t \mu_{\mathbf{X}} \right\} \exp \left\{ -\frac{1}{2} \|\mathbf{H}^t \omega\|^2 \right\}$$

or

$$\Phi_X(\omega) = \exp \left\{ i\omega^t \mu_{\mathbf{X}} - \frac{1}{2} \omega^t \mathbf{K}_{\mathbf{X}} \omega \right\}.$$

3.4 Density Function of a Gaussian Random Vector

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{a}.$$

Assume \mathbf{H} is invertible. If

$$\mathbf{X} = g(\mathbf{W}) = (g_1(\mathbf{W}), \dots, g_n(\mathbf{W}))^t$$

then

$$f_{\mathbf{X}}(\mathbf{x}) = \sum_k \frac{f_{\mathbf{W}}(\mathbf{y}_k)}{|J(\mathbf{y}_k)|}$$

where

$$g_i(\mathbf{y}_k) = x_i, \quad i = 1, \dots, n, \quad \mathbf{y}_k = (y_{k,1}, \dots, y_{k,n})^t, \quad \mathbf{x} = (x_1, \dots, x_n)^t$$

and J denotes the Jacobian of the transformation.

For us

$$\mathbf{X} = \mathbf{H}\mathbf{W} + \mathbf{a} = g(\mathbf{W}).$$

So we solve

$$\mathbf{x} = \mathbf{H}\mathbf{w} + \mathbf{a} \Rightarrow \mathbf{w} = \mathbf{H}^{-1}(\mathbf{x} - \mathbf{a}).$$

$$J(\mathbf{w}) = \mathbf{H}$$

so

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\det(\mathbf{H})} f_{\mathbf{W}}(\mathbf{H}^{-1}(\mathbf{x} - \mathbf{a}))$$

$$= \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2} \|\mathbf{H}^{-1}(\mathbf{x} - \mathbf{a})\|^2 \right\} \cdot \frac{1}{\det(\mathbf{H})}.$$

Now

$$\mathbf{K}_{\mathbf{X}} = \mathbf{H}\mathbf{H}^t \Rightarrow \det(\mathbf{K}_{\mathbf{X}}) = (\det(\mathbf{H}))^2.$$

So

$$\det(\mathbf{H}) = \sqrt{\det(\mathbf{K}_{\mathbf{X}})}.$$

Thus,

$$\begin{aligned} f_X(x) &= \frac{1}{(2\pi)^{n/2} \sqrt{\det(\mathbf{K}_{\mathbf{X}})}} \exp \left(-\frac{1}{2} (\mathbf{H}^{-1}(\mathbf{x} - \mathbf{a}))^\dagger (\mathbf{H}^{-1}(\mathbf{x} - \mathbf{a})) \right) \\ &= \frac{1}{(2\pi)^{n/2} \sqrt{\det(\mathbf{K}_{\mathbf{X}})}} \exp \left(-\frac{1}{2} (\mathbf{x} - \mathbf{a})^\dagger (\mathbf{H}^{-1})^\dagger \mathbf{H}^{-1} (\mathbf{x} - \mathbf{a}) \right) \end{aligned}$$

which becomes

$$f_X(x) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(\mathbf{K}_{\mathbf{X}})}} \exp \left(-\frac{1}{2} (\mathbf{x} - \mu_{\mathbf{X}})^\dagger \mathbf{K}_{\mathbf{X}}^{-1} (\mathbf{x} - \mu_{\mathbf{X}}) \right).$$