EE 562a Midterm Solution

July 1, 2009

Inst: Dr. C.W. Walker

Problem	Points	Score
1	**14**	
2	**14**	
3	**12**	
4	**15**	
5	**15**	
6	**15**	
7	**15**	
Total	**100**	

Instructions and Information:

1) Print your name at the top of the page and indicate whether or not you are a DEN (off-campus) student.
2) Make sure your exam has 7 problems.
3) This is a closed book, closed notes exam. You may use one 8 ½ x 11 in. sheet of notes (front and back). You may use a calculator but not a computer. **You have 2 hours to take this exam.**
4) Partial credit will be given but you must show your work to receive any credit.
5) **Circle or box your final answers**.

Problem 1. Let $\mathbf{W}$ be a white random vector with

$$\mu_{\mathbf{W}} = (0 \quad 0)^t, \quad \mathbf{K}_{\mathbf{W}} = \mathbf{I}.$$

Let

$$\mathbf{X} = \mathbf{H}\mathbf{W} + \mathbf{c}.$$

Find $\mathbf{c}$ and a causal matrix $\mathbf{H}$ with real and positive entries that produces

$$\mu_{\mathbf{X}} = [1 \quad 1]^t, \quad \mathbf{K}_{\mathbf{X}} = \begin{bmatrix} 16 & 2 \\ 2 & 8 \end{bmatrix}.$$

Solution. We have

$$\begin{bmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} \\ 0 & h_{22} \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 2 & 8 \end{bmatrix}$$

so

$$h_{11}^2 = 16 \Rightarrow h_{11} = 4$$

$$h_{11}h_{21} = 2 \Rightarrow h_{21} = \frac{1}{2}$$

$$h_{21}^2 + h_{22}^2 = 8 \Rightarrow h_{22} = \frac{\sqrt{31}}{2}$$

so

$$\mathbf{H} = \begin{bmatrix} 4 & 0 \\ \frac{1}{2} & \frac{\sqrt{31}}{2} \end{bmatrix}$$

and

$$\mathbf{c} = [1 \quad 1]^t.$$

Problem 2. Let $\mathbf{X}(u)$ be a random vector with correlation matrix $\mathbf{R_X}$ where

$$\mathbf{R_X} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

This correlaiton matrix has eigenvectors $\mathbf{e}_1 = \frac{1}{\sqrt{2}}[1 \quad 1]^t$ and $\mathbf{e}_2 = \frac{1}{\sqrt{2}}[1 \quad -1]^t$. Let

$$Y_i(u) = \mathbf{e}_i^\dagger \mathbf{X}(u), \quad i = 1, 2.$$

a. Compute the numerical value of $E[|Y_1(u)|^2]$.

Solution.

$$\begin{aligned} E[|Y_1(u)|^2] &= E[Y_1(u)Y_1^*(u)] = E[\mathbf{e}_1^\dagger \mathbf{X}(u)\mathbf{X}^\dagger(u)\mathbf{e}_1] \\ &= \mathbf{e}_1^\dagger \mathbf{R_X} \mathbf{e}_1 \\ &= \frac{1}{\sqrt{2}}[1 \quad 1] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3. \end{aligned}$$

b. Compute the numerical value of $E[Y_1(u)Y_2(u)^*]$.

Solution.

$$\begin{aligned} E[Y_1(u)Y_2(u)^*] &= E[\mathbf{e}_1^\dagger \mathbf{X}(u)\mathbf{X}^\dagger(u)\mathbf{e}_2] \\ &= \mathbf{e}_1^\dagger \mathbf{R_X} \mathbf{e}_2 \\ &= \frac{1}{\sqrt{2}}[1 \quad 1] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0. \end{aligned}$$

Problem 3. Suppose u is selected uniformly in the interval [0, 1]. For each of the following state whether the sequence of random variables converges surely to a constant, almost surely to a constant or neither of these. Note that in this problem we are only concerned about convergence to a constant or not. Please circle your answers.

a. $X_n(u) = nu^2$.

Solution. $X_n(u)$ tends to infinity for all $u \neq 0$ so it does not converge to a constant.

b. $Y_n(u) = 1 - \dfrac{n^2 - n}{n^2} u$.

Solution. $Y_n(u)$ converges to the random variable $1 - U(u)$ so it does not converge to a constant.

c. $V_n(u) = u^{1/n}$.

Solution. $V_n(u)$ converges almost surely to 1.

d. $\log_{10}(1 + u^n)$.

Solution. This converges almost surely to 0.

Problem 4. We are given an observation of X and we must decide between two hypotheses:

$$H_0: \quad X = N$$

$$H_1: \quad X = S + N$$

where, N is a continuous random variable with density

$$f_N(x) = \begin{cases} 2e^{-2x}, & 0 \le x < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

and $S = 6$. Find a threshold T (to 3 decimal places) so that the probability of type I error is 10^{-4}.

Solution. We solve

$$\int_T^\infty 2e^{-2x} dx = 10^{-4}$$

for T to get

$$T = -\frac{1}{2} \ln \left(10^{-4}\right) = 4.605.$$

Problem 5. Let $\mathbf{X}$(u) be a Gaussian random vector with density $\mathbf{X} \sim N(\mu_{\mathbf{X}}, \mathbf{K_X})$ where

$$\mu_{\mathbf{X}} = [1 \quad 1 \quad 1]^t, \quad \mathbf{K_X} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Let

$$\mathbf{Y}(u) = \mathbf{G}\mathbf{X}(u) + \mathbf{a}$$

where

$$\mathbf{G} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

and

$$\mathbf{a} = [3 \quad 2]^t.$$

Find the density function, $f_{\mathbf{Y}}(\mathbf{y})$, of $\mathbf{Y}$(u). Note your answer should be a function of $\mathbf{y}$ only, i.e., your answer should be in terms of numerical values and $\mathbf{y} = [y_1 \quad y_2]^t$. You can write your final answer using variables but the variables should be defined numerically in your solution.

Solution.

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{det(\mathbf{K_Y})}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mu_{\mathbf{Y}})^{\dagger}\mathbf{K_Y^{-1}}(\mathbf{y} - \mu_{\mathbf{Y}})\right)$$

where

$$det(\mathbf{K_Y}) = det(\mathbf{G}\mathbf{K_X}\mathbf{G}^{\dagger}) = \begin{vmatrix} 18 & 26 \\ 26 & 46 \end{vmatrix} = 152$$

$$\mu_{\mathbf{Y}} = \mathbf{G}\mu_{\mathbf{X}} + \mathbf{a} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\mathbf{K_Y^{-1}} = \frac{1}{152}\begin{bmatrix} 46 & -26 \\ -26 & 18 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Problem 6. Design a correlation detector for deciding whether H_0 or H_1 is true where

$$H_i: \ \mathbf{X}(u) = \mathbf{S}_i + \mathbf{N}(u), \quad i = 0, 1$$

and

$$\mathbf{S}_i = (-1)^i \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$E[\mathbf{N}(u)] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{K_N} = \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix}.$$

Indicate the decision regions on a graph.

Solution. We find

$$\mathbf{K}_\mathbf{N}^{-1} = \frac{1}{63}\begin{bmatrix} 8 & 1 \\ 1 & 8 \end{bmatrix}, \ \mathbf{S}_1 - \mathbf{S}_0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix},$$

$$T = \frac{\mathbf{S}_1^\dagger \mathbf{K}_\mathbf{N}^{-1} \mathbf{S}_1 - \mathbf{S}_0^\dagger \mathbf{K}_\mathbf{N}^{-1} \mathbf{S}_0}{2} = 0.$$

$$< \mathbf{x} - \mu_N, \mathbf{K}_\mathbf{N}^{-1}(\mathbf{S}_1 - \mathbf{S}_0) > < 0 \ \text{ for } H_0$$

or

$$[x_1 - 1, \ x_2 - 1]\mathbf{K}_\mathbf{N}^{-1}(\mathbf{S}_1 - \mathbf{S}_0) \ < 0 \ \text{ for } H_0$$

which yields

$$8x_1 + x_2 > 9 \ \text{ for } H_0.$$

The decision boundary is a straight line intersecting the x_1-axis at 9/8 and intersection the x_2-axis at 9.

Problem 7. Let $Z(n)$ be an i.i.d. Bernoulli sequence where

$$P(Z(n) = 1) = 1/2, \quad P(Z(n) = -1) = 1/2.$$

a. For $n \geq 1$ let

$$X(n) = Z(n) + \sum_{k=0}^{n-1} X(k)$$

where we take $Z(0) = 0 = X(0)$. Find $\mathbf{R_X}(n, m)$.

Solution. Observe that

$$X(n) = Z(n) + \sum_{k=1}^{n-1} 2^{n-1-k} Z(k).$$

Then

$$\mathbf{R_X}(n, m) = E\left[\left(Z(n) + \sum_{i=1}^{n-1} 2^{n-1-i} Z(i)\right)\left(Z(m) + \sum_{k=1}^{m-1} 2^{m-1-k} Z(k)\right)\right]$$

We use $E[Z(n)] = 0$ and $E[Z(n)^2] = 1$ and $E[Z(i)Z(k)] = 0$ for $i \neq k$.

Case 1: $n = m$.

$$\begin{aligned}
\mathbf{R_X}(n, n) &= E\left[\left(Z(n) + \sum_{i=1}^{n-1} 2^{n-1-i} Z(i)\right)\left(Z(n) + \sum_{k=1}^{n-1} 2^{n-1-k} Z(k)\right)\right] \\
&= 1 + \sum_{k=1}^{n-1} 4^{n-1-k} = \frac{4^{n-1} + 2}{3}.
\end{aligned}$$

Case 2: $n > m$.

$$\begin{aligned}
\mathbf{R_X}(n, m) &= E\left[\sum_{i=1}^{n-1} 2^{n-1-i} Z(i) Z(m)\right] + E\left[\sum_{i=1}^{m-1} 2^{n-1-i} 2^{m-1-i} Z(i) Z(i)\right] \\
&= 2^{n-1-m} + \sum_{i=1}^{m-1} 2^{n-1-i} 2^{m-1-i} = 2^{n-1-m} + 2^{m+n} \frac{\frac{1}{4} - \left(\frac{1}{4}\right)^m}{3}.
\end{aligned}$$

Case 3: $n < m$.

$$\mathbf{R_X}(n, m) = 2^{m-1-n} + 2^{m+n} \frac{\frac{1}{4} - \left(\frac{1}{4}\right)^n}{3}.$$

b. Let

$$S_n = \sum_{i=1}^{n} Z(i).$$

For any positive integer N, show

$$P\left(\max_{1\le j\le n} S_j \ge N,\ S_n < N\right) = P(S_n > N).$$

Solution. Let T denote the smallest integer j in $[1, n]$ such that $S_j = N$ if such a j exists. Otherwise, let $T = n + 1$.

$$\begin{aligned}
P\left(\max_{1\le j\le n} S_j \ge N,\ S_n < N\right) &= P(T \le n,\ S_n < N) = P(T < n,\ S_n < N) \\
&= \sum_{k=1}^{n-1} P\left(T = k, \sum_{i=k+1}^{n} Z_i < 0\right) \\
&= \sum_{k=1}^{n-1} \left[P(T = k)P\left(\sum_{i=k+1}^{n} Z_i < 0\right)\right] \\
&= \sum_{k=1}^{n-1} \left[P(T = k)P\left(\sum_{i=k+1}^{n} Z_i > 0\right)\right] \\
&= \sum_{k=1}^{n-1} P(T = k,\ S_n > N) \\
&= P(T < n,\ S_n > N) \\
&= P(S_n > N).
\end{aligned}$$