

25.0 Bandpass Systems

25.1 Representations of Bandpass Systems

Let $s(t)$ be real-valued.

$$s(t) = a(t) \cos[2\pi f_c t + \theta(t)] \quad (1)$$

$a(t) = \text{amplitude (or envelope) of } s(t)$

$\theta(t) = \text{phase of } s(t)$

$f_c = \text{carrier frequency of } s(t)$

If bandwidth is much smaller than f_c , we have a bandpass system.

$$\begin{aligned} s(t) &= a(t) \cos(\theta(t)) \cos(2\pi f_c t) - a(t) \sin(\theta(t)) \sin(2\pi f_c t) \\ &= x(t) \cos(2\pi f_c t) - y(t) \sin(2\pi f_c t) \end{aligned} \quad (2)$$

$x(t) = a(t) \cos(\theta(t)) \longrightarrow \text{in phase component}$

$y(t) = a(t) \sin(\theta(t)) \longrightarrow \text{quadrature component}$

$x(t)$ and $y(t)$ are low-pass signals, since their frequency component is concentrated around $f = 0$.

Let

$$\begin{aligned} u(t) &= a(t)e^{i\theta(t)} \\ &= x(t) + iy(t) \end{aligned}$$

Then,

$$s(t) = \text{Re}\{u(t)e^{i2\pi f_c t}\} \quad (3)$$

So, $s(t)$ has the 3 representations shown above in (1), (2) and (3)

$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft} dt \\ &= \int_{-\infty}^{\infty} \{\text{Re}[u(t)e^{i2\pi f_c t}]\} e^{-i2\pi ft} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} [u(t)e^{i2\pi f_c t} + u^*(t)e^{-i2\pi f_c t}] e^{-i2\pi ft} dt \end{aligned}$$

$$= \frac{1}{2}[U(f - f_c) + U^*(-f - f_c)]$$

where, $u(t) \xleftrightarrow{F.T.} U(f)$

Since frequency content of $s(t)$ is concentrated around f_c , we see that the frequency content of $u(t)$ is around $f = 0$. So, the complex valued waveform $u(t)$ is a low-pass signal waveform and is called the equivalent low-pass signal

The energy in $s(t)$ is

$$\begin{aligned} \xi &= \int_{-\infty}^{\infty} s^2(t) dt \\ &= \int_{-\infty}^{\infty} \{Re[u(t)e^{i2\pi f_c t}]\}^2 dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} |u(t)|^2 dt + \underbrace{\frac{1}{2} \int_{-\infty}^{\infty} |u(t)|^2 \cos[4\pi f_c t + 2\theta(t)] dt}_{\text{small compared to the 1st integral}} \end{aligned}$$

So,

$$\xi \approx \frac{1}{2} \int_{-\infty}^{\infty} |u(t)|^2 dt$$

where, $|u(t)| = a(t)$, the envelope.

25.2 Representations of Linear Bandpass Systems

Here $h(t)$ is real, so

$$H^*(-f) = H(f)$$

Define,

$$C(f - f_c) = \begin{cases} H(f), & f > 0 \\ 0, & f < 0. \end{cases}$$

Then,

$$C^*(-f - f_c) = \begin{cases} 0, & f > 0 \\ H^*(-f), & f < 0. \end{cases}$$

So,

$$H(f) = C(f - f_c) + C^*(-f - f_c)$$

$$\begin{aligned}\implies h(t) &= c(t)e^{i2\pi f_c t} + c^*(t)e^{-i2\pi f_c t} \\ &= 2\text{Re}[c(t)e^{i2\pi f_c t}]\end{aligned}$$

Here, $c(t)$ is the impulse response of the equivalent low-pass system and is complex.

A filter that is encountered in the generation of single-sideband signal has the impulse response,

$$\begin{aligned}h(t) &= \frac{1}{\pi t} \\ \implies H(f) &= \begin{cases} -i, & f > 0 \\ i, & f < 0. \end{cases}\end{aligned}$$

$H(f)$ represents an all-pass filter which introduces a -90° phase shift for $f < 0$.

The output is (for input $s(t)$)

$$r(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} d\tau$$

This is called a Hilbert transform \longrightarrow output = $\hat{s}(t)$

25.3 Response of a Bandpass System to a Bandpass Signal

So far we have seen that a narrowband bandpass signal and system can be represented by equivalent low-pass signals and systems.

We want to look at the output,

$$\begin{aligned}s(t) &= \text{Re}[u(t)e^{i2\pi f_c t}] \\ h(t) &= 2\text{Re}[c(t)e^{i2\pi f_c t}] \\ r(t) &= \text{Re}[v(t)e^{i2\pi f_c t}], \quad \text{some } v(t)\end{aligned}$$

where,

$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t - \tau)d\tau$$

$$R(f) = S(f)H(f)$$

Or,

$$R(f) = \frac{1}{2}[U(f - f_c) + U^*(-f - f_c)][C(f - f_c) + C^*(-f - f_c)]$$

where $s(t)$ is a narrowband signal and $h(t)$ is the impulse response of a narrowband system

$$U(f - f_c) \approx 0 \text{ for } f < 0 \text{ and } C(f - f_c) = 0 \text{ for } f < 0.$$

So,

$$U(f - f_c)C^*(-f - f_c) = 0$$

and,

$$U^*(-f - f_c)C(f - f_c) = 0$$

So,

$$\begin{aligned} R(f) &= \frac{1}{2}[U(f - f_c)C(f - f_c) + U^*(-f - f_c)C^*(-f - f_c)] \\ &= \frac{1}{2}[V(f - f_c) + V^*(-f - f_c)] \end{aligned}$$

where $V(f) = U(f)C(f)$ is the output spectrum of the equivalent low-pass system excited by the equivalent low-pass signal.

So,

$$v(t) = u(t) * c(t)$$

or,

$$v(t) = \int_{-\infty}^{\infty} u(\tau)c(t - \tau)d\tau$$

These relationships between bandpass and equivalent low-pass signals allow us to ignore any linear frequency translations encountered in the modulation of a signal for the purpose of matching its spectral content to the frequency allocation of a particular channel.

25.4 Representations of Bandpass Stationary Stochastic Processes

Let $n(t)$ be a WSS stochastic process with zero mean.

$$\begin{aligned}
 n(t) &= a(t) \cos[2\pi f_c t + \theta(t)] \\
 &= x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \\
 &= \operatorname{Re}[z(t)e^{i2\pi f_c t}] \\
 a(t) &= \text{envelope} \\
 z(t) &= x(t) + iy(t) \text{ (complex envelope)} \\
 E[n(t)] &= 0 \implies E[x(t)] = E[y(t)] = 0
 \end{aligned}$$

Claim

$$\begin{aligned}
 R_X(\tau) &= R_Y(\tau) \\
 R_{XY}(\tau) &= -R_{YX}(\tau)
 \end{aligned}$$

Proof

$$\begin{aligned}
 R_n(\tau) &= E[n(t)n(t-\tau)] \\
 &= E[(x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t)(x(t-\tau) \cos 2\pi f_c(t-\tau) - y(t-\tau) \sin 2\pi f_c(t-\tau))] \\
 &= R_X(\tau) \cos 2\pi f_c t \cos 2\pi f_c(t-\tau) \\
 &\quad + R_Y(\tau) \sin 2\pi f_c t \sin 2\pi f_c(t-\tau) \\
 &\quad - R_{YX}(\tau) \sin 2\pi f_c t \cos 2\pi f_c(t-\tau) \\
 &\quad - R_{XY}(\tau) \cos 2\pi f_c t \sin 2\pi f_c(t-\tau)
 \end{aligned}$$

Use

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$R_n(\tau) = \frac{1}{2}[R_X(\tau) + R_Y(\tau)] \cos 2\pi f_c \tau$$

$$\begin{aligned}
& +\frac{1}{2}[R_X(\tau) - R_Y(\tau)] \cos 2\pi f_c(2t - \tau) \\
& -\frac{1}{2}[R_{YX}(\tau) + R_{XY}(\tau)] \sin 2\pi f_c\tau \\
& -\frac{1}{2}[R_{YX}(\tau) + R_{XY}(\tau)] \sin 2\pi f_c(2t - \tau)
\end{aligned}$$

RHS must be independent of t for $n(t)$ to be WSS.

$$\implies R_X(\tau) = R_Y(\tau)$$

$$R_{XY}(\tau) = -R_{YX}(\tau)$$

Thus,

$$R_n(\tau) = R_X(\tau) \cos 2\pi f_c\tau - R_{YX}(\tau) \sin 2\pi f_c\tau$$

The autocorrelation function of the equivalent low-pass process

$$z(t) = x(t) + iy(t)$$

is defined as

$$\begin{aligned}
R_Z(\tau) &= \frac{1}{2}E[z(t)z^*(t + \tau)] \\
&= \frac{1}{2}[R_X(\tau) + R_Y(\tau) - iR_{XY}(\tau) + iR_{YX}(\tau)] \\
&= R_X(\tau) + iR_{YX}(\tau)
\end{aligned}$$

So,

$$R_n(\tau) = \text{Re}[R_Z(\tau)e^{i2\pi f_c\tau}]$$

Thus, the autocorrelation function $R_n(\tau)$ of the bandpass stochastic process is determined from $R_Z(\tau)$, the autocorrelation function of the equivalent low-pass process $z(t)$ and the carrier frequency f_c .

Now,

$$\begin{aligned}
S_n(f) &= \int_{-\infty}^{\infty} \{\text{Re}[R_Z(\tau)e^{i2\pi f_c\tau}]\} e^{-i2\pi f_c\tau} d\tau \\
&= \frac{1}{2}[S_Z(f - f_c) + S_Z(-f - f_c)]
\end{aligned}$$

25.4.1 Properties of the In-Phase and Quadrature Components

Since

$$R_{XY}(\tau) = -R_{YX}(\tau)$$

and

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

We get

$$\begin{aligned} R_{XY}(\tau) &= -R_{XY}(-\tau) \\ \implies R_{XY}(\tau) &\text{ is an odd function of } \tau \end{aligned}$$

So,

$$R_{XY}(0) = 0 \implies x(t) \text{ and } y(t) \text{ are uncorrelated for } \tau = 0.$$

If $n(t)$ is a Gaussian process, then $x(t + \tau)$ and $y(t)$ are jointly Gaussian and for $\tau = 0$ they are uncorrelated \implies independent.

So, in this case their joint pdf is

$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

where $\sigma^2 = R_X(0) = R_Y(0) = R_n(0)$

25.4.2 Representation of White Noise

The noise resulting from passing white noise through a spectrally flat (ideal) bandpass filter is termed bandpass white noise.

The equivalent low-pass noise $z(t)$ has

$$\begin{aligned} S_Z(f) &= \begin{cases} N_o, & |f| \leq \frac{B}{2} \\ 0, & |f| > \frac{B}{2}. \end{cases} \\ \implies R_Z(\tau) &= N_o \frac{\sin \pi B \tau}{\pi \tau} \end{aligned}$$

As $B \rightarrow \infty$

$$R_Z(\tau) \longrightarrow N_o \delta(\tau)$$

The power spectral density for white noise and bandpass white noise is symmetric about $f = 0$, so $R_{YX}(\tau) = 0 \quad \forall \tau$.

Thus,

$$R_Z(\tau) = R_X(\tau) = R_Y(\tau)$$

$\implies x(t)$ and $y(t)$ are uncorrelated for all time shifts τ and the autocorrelation functions of $z(t)$, $x(t)$ and $y(t)$ are all equal.