

## 24.0 LMMSE Estimation and K-L Expansions

Let

$$X(u, t) = S(u, t) + W(u, t)$$

where  $S(u, t)$  is a mean zero Gaussian process and  $W(u, t)$  is mean zero white noise. Furthermore,  $S(u, t)$  is independent of  $W(u, t)$ . We observe  $X(u, t)$  in the interval  $[0, T]$ . We want to estimate  $S$  so that

$$E \left[ \int_0^T |S(u, t) - \hat{S}(u, t)|^2 dt \right]$$

is minimized. Write

$$S(u, t) = \sum_{n=1}^{\infty} S_n(u) \phi_n(t)$$

where  $\phi_n(t)$  is a complete set of orthonormal eigenfunctions. Also,

$$W(u, t) = \sum_{n=1}^{\infty} W_n(u) \phi_n(t).$$

So

$$X(u, t) = \sum_{n=1}^{\infty} X_n(u) \phi_n(t)$$

where

$$X_n(u) = S_n(u) + W_n(u).$$

We know

$$\hat{S}_n(u) = \frac{\sigma_{S_n}^2}{\sigma_{S_n}^2 + \sigma_W^2} X_n(u).$$

Thus

$$\hat{S}(u, t) = \sum_{n=1}^{\infty} \hat{S}_n(u) \phi_n(t).$$

Note

$$E \left[ |S_n(u) - \hat{S}_n(u)|^2 \right] = \frac{\sigma_{S_n}^2 \sigma_W^2}{\sigma_{S_n}^2 + \sigma_W^2}$$

and the total mean square error is

$$MSE = \sum_{n=1}^{\infty} \frac{\sigma_{S_n}^2 \sigma_W^2}{\sigma_{S_n}^2 + \sigma_W^2}.$$