

22.0 Karhunen-Loeve Expansion

We will assume zero mean in this section.

Mercer's Theorem: If $K(t, s)$ is a continuous positive semi-definite function, then if $\{\lambda_n, \phi_n(t)\}$ are a complete set of orthonormal solutions to

$$\int_a^b K(t, s)\phi_n(s)ds = \lambda_n\phi_n(t), \quad a \leq t \leq b$$

then

$$K(t, s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t) \phi_n^*(s) \quad \forall t, s \in [a, b].$$

Here $\lambda_n = E[|X_n|^2]$.

Note for vectors we have

$$K_X = \sum_{n=1}^N \lambda_n e_n e_n^{*T} = E \Lambda E^{*T}.$$

Note that if $a = E^{*T} X$ then a_1, \dots, a_N are uncorrelated random variables with $E[a_i^2] = \lambda_i$.

K-L Theorem: If $X(t)$ is a zero-mean 2nd order random process on $[-T/2, T/2]$ with continuous covariance $K_X(t_1, t_2)$ then

$$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t), \quad \forall |t| \leq T/2 \quad (1)$$

where

$$X_n = \int_{-T/2}^{T/2} X(t) \phi_n^*(t) dt \quad (2)$$

and $\{\phi_n(t)\}$ are an orthonormal set of solutions to

$$\int_{-T/2}^{T/2} K_X(t_1, t_2) \phi_n(t_2) dt_2 = \lambda_n \phi_n(t_1), \quad \forall |t_1| \leq T/2 \quad (3)$$

and

$$E[X_n X_m^*] = \lambda_n \delta(n - m). \quad (4)$$

Proof:

1, 2, 4 \Rightarrow 3.

By (1)

$$E[X(t_1)X_n^*] = \sum_{n=1}^{\infty} E[X_n X_n^*] \phi_n(t_1) \quad (MS)$$

and by (4)

$$= E[|X_n|^2] \phi_n(t_1) = \lambda_n \phi_n(t_1)$$

and by (2)

$$= E[X(t_1) \int_{-T/2}^{T/2} X^*(t_2) \phi_n(t_2) dt_2].$$

So

$$\lambda_n \phi_n(t) = \int_{-T/2}^{T/2} K_X(t_1, t_2) \phi_n(t_2) dt_2.$$

1, 2, 3 \Rightarrow 4.

$$E[X(t)X_m^*] = \lambda_m \phi_m(t).$$

$$\begin{aligned} E[X_n X_m^*] &= E \left[\int_{-T/2}^{T/2} X(t) \phi_n^*(t) dt \cdot X_m^* \right] \\ &= \int_{-T/2}^{T/2} \lambda_m \phi_m(t) \phi_n^*(t) dt \\ &= \lambda_n \delta(n - m). \end{aligned}$$

Now let

$$\hat{X}(t) = \sum_{n=1}^{\infty} X_n \phi_n(t)$$

where

$$X_n = \int_{-T/2}^{T/2} X(t) \phi_n^*(t) dt.$$

Then

$$E[|X(t) - \hat{X}(t)|^2] = 0.$$

So

$$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \quad (MS).$$

Example: White Noise.

$$K_X(t_1, t_2) = \sigma^2 \delta(t_1 - t_2).$$

$$\int_{-T/2}^{T/2} \sigma^2 \delta(t_1 - t_2) \phi(t_2) dt_2 = \lambda \phi(t_1).$$

$$\sigma^2 \phi(t_1) = \lambda \phi(t_1)$$

so

$$\lambda = \sigma^2.$$

Therefore, a basis for white noise $\{\phi_n(t)\}$ can be any complete set of orthonormal functions with corresponding eigenvalues $\lambda_n = \sigma^2$.

Example: Bandlimited Noise. Assume zero-mean and WSS.

$$S_X(f) = \int_{-\infty}^{\infty} K_X(\tau) \exp(-i2\pi f\tau) d\tau.$$

$X(t)$ is bandlimited if $S_X(f) = 0$ for $|f| > f_{max}$. Suppose $S_X(f) = 1$ for $|f| \leq f_{max}$. Then

$$R(\tau) = \int_{-f_{max}}^{f_{max}} 1 \cdot \exp(i2\pi f\tau) df = \frac{\sin 2\pi f_{max}\tau}{\pi\tau}.$$

$$\int_{-T/2}^{T/2} \frac{\sin 2\pi f_{max}(t_1 - t_2)}{\pi(t_1 - t_2)} \phi_n(t_2) dt_2 = \lambda_n \phi_n(t_1).$$

Prolate spheroidal functions solve this equation for ϕ_n .