

## 20.0 Mean Square Calculus

### 20.1 Mean Square Continuity

Imposing continuity on all sample functions is too restrictive in general, therefore we use mean square continuity.

**Definition:** The random process  $X(t) = X(u, t)$  is *mean square continuous* at  $t = t_0$  if

$$\lim_{t \rightarrow t_0} E [ |X(t) - X(t_0)|^2 ] = 0.$$

If it holds for all  $t_0 \in T$ ,  $X(t)$  is *mean square continuous*.

**Theorem:**  $X(t)$  is mean square continuous at  $t_0$  if and only if  $R_X(t_1, t_2)$  is continuous at the point  $t_1 = t_2 = t_0$ .

**Proof:** “ $\Leftarrow$ ”.

$$E [ |X(t) - X(t_0)|^2 ] = R_X(t, t) - R_x(t_0, t) - R_x(t, t_0) + R_X(t_0, t_0) = 0$$

as  $t \rightarrow t_0$ .

“ $\Leftarrow$ ”. If the *LHS*  $\rightarrow 0$  then the *RHS*  $\rightarrow 0$  so  $R_X(t_1, t_2)$  must be continuous at  $t_1 = t_2 = t_0$  as  $t \rightarrow t_0$ .

**Note:** If  $X(t)$  is WSS then  $X(t)$  is mean square continuous if and only if  $R_X(\tau)$  is continuous at  $\tau = 0$ .

**Theorem:** The mean function of a mean square continuous random process is a continuous function.

**Proof:**

$$(\mu_X(t) - \mu_X(t_0))^2 = [E (X(t) - X(t_0))]^2 \leq E [(X(t) - X(t_0))^2].$$

As  $t \rightarrow t_0$  the *RHS*  $= 0 \Rightarrow$  *LHS*  $= 0$

$$\Rightarrow \lim_{t \rightarrow t_0} (\mu_X(t) - \mu_X(t_0))^2 = 0$$

so

$$\mu_X(t) \rightarrow \mu_X(t_0)$$

i.e.,

$$\lim_{t \rightarrow t_0} E[X(t)] = E\left[\lim_{t \rightarrow t_0} X(t)\right].$$

**Example:** Wiener process.

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2).$$

As

$$(t_1, t_2) \rightarrow (t, t)$$

we have

$$\alpha \min(t_1, t_2) \rightarrow \alpha t.$$

Is this M.S. continuous? Consider

$$|\alpha \min(t_1, t_2) - \alpha t|.$$

Let  $t_1 = t + \epsilon_1$ ,  $\epsilon_1 > 0$ ,  $t_2 = t + \epsilon_2$ ,  $\epsilon_2 > 0$ . Then

$$\alpha |\min(t + \epsilon_1, t + \epsilon_2) - t| \leq \alpha \max(\epsilon_1, \epsilon_2) \rightarrow 0 \text{ as } \epsilon_1, \epsilon_2 \rightarrow 0$$

$\Rightarrow R_X(t_1, t_2)$  is continuous at this point so the Wiener process is M.S. continuous.

## 20.2 Mean Square Derivatives

**Theorem:** The M.S. derivative exists at  $t$  if and only if

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$$

exists when  $(t_1, t_2) = (t, t)$ .

**Proof:** Since  $X'(t)$  is not known we will use the Cauchy criterion.

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} E \left[ \left| \frac{X(t + \epsilon_1) - X(t)}{\epsilon_1} - \frac{X(t + \epsilon_2) - X(t)}{\epsilon_2} \right|^2 \right]$$

$$\begin{aligned}
&= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} E \left[ \left| \frac{X(t + \epsilon_1) - X(t)}{\epsilon_1} \right|^2 \right] + E \left[ \left| \frac{X(t + \epsilon_2) - X(t)}{\epsilon_2} \right|^2 \right] \\
&\quad - 2E \left[ \left( \frac{X(t + \epsilon_1) - X(t)}{\epsilon_1} \right) \left( \frac{X^*(t + \epsilon_2) - X^*(t)}{\epsilon_2} \right) \right].
\end{aligned}$$

The first two terms of this last expression are of the form

$$E \left[ \left| \frac{X(t + \epsilon) - X(t)}{\epsilon} \right|^2 \right] = \frac{1}{\epsilon} \left[ \frac{R_X(t + \epsilon, t + \epsilon) - R_X(t, t + \epsilon)}{\epsilon} - \frac{R_X(t + \epsilon, t) - R_X(t, t)}{\epsilon} \right].$$

This converges to

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t)$$

if the second mixed partial derivative w.r.t.  $t_1$  and  $t_2$  exists at the point  $(t, t)$ .

The last term in our expression above is

$$\begin{aligned}
&-2E \left[ \left( \frac{X(t + \epsilon_1) - X(t)}{\epsilon_1} \right) \left( \frac{X^*(t + \epsilon_2) - X^*(t)}{\epsilon_2} \right) \right] \\
&= \frac{-2}{\epsilon_1} \left[ \frac{R_X(t + \epsilon_1, t + \epsilon_2) - R_X(t + \epsilon_1, t)}{\epsilon_2} - \frac{R_X(t, t + \epsilon_2) - R_X(t, t)}{\epsilon_2} \right]
\end{aligned}$$

which converges to

$$-2 \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t).$$

So our expression becomes

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t) + \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t) - 2 \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t) = 0.$$

Thus the M.S. derivative exists.

Also, the M.S. derivative of a W.S. stationary R.P.  $X(t)$  exists if  $R_X(\tau)$  has derivatives up to order two at  $\tau = 0$ .

**Example:** Let

$$X(u, t) = A(u) \cos(2\pi t).$$

Then

$$R_X(t_1, t_2) = E[A^2(u) \cos(2\pi t_1) \cos(2\pi t_2)].$$

$$\left. \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t, t) \right|_{t_1=t_2=t} = 4\pi^2 E[A^2(u)] \sin^2(2\pi t)$$

$\Rightarrow X(t)$  has a M.S. derivative at every point  $t$ .

**Example:**

$$R_X(\tau) = \sigma^2 \exp(-\alpha^2 \tau^2).$$

$$\frac{d}{d\tau} R_X(\tau) = -2\alpha^2 \sigma^2 \tau \exp(-\alpha^2 \tau^2).$$

$$\frac{d^2}{d\tau^2} R_X(\tau) = 4\alpha^4 \sigma^2 \tau \exp(-\alpha^2 \tau^2) - 2\alpha^2 \sigma^2 \exp(-\alpha^2 \tau^2).$$

$$R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau) = (1 - 2\alpha^2 \tau^2) 2\alpha^2 \sigma^2 \exp(-\alpha^2 \tau^2).$$

**Example:** Wiener process.

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2).$$

$$\frac{\partial R_X(t_1, t_2)}{\partial t_1} = \alpha u(t_2 - t_1).$$

But the derivative of a step function does not exist at its point of discontinuity, so the second mixed partial derivative does not exist. Hence the Wiener process does not have a M.S. derivative at any point. Instead we use a generalized derivative as

$$\frac{\partial R_X(t_1, t_2)}{\partial t_1 \partial t_2} = \alpha \delta(t_2 - t_1)$$

$$R_{X'}(t_1, t_2) = \alpha \delta(t_2 - t_1)$$

or

$$R_{X'}(\tau) = \alpha \delta(\tau)$$

which implies a flat spectral density. So the generalized M.S. derivative of the Wiener process is white Gaussian noise.

## 20.3 Mean Square Integrals

**Example:** White Gaussian noise.

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sigma^2 \delta(t_1 - t_2).$$

Let

$$Y(t) = \int_0^t X(\tau) d\tau.$$

$Y(t)$  is a stochastic process since for a fixed  $t = t_0$ ,  $Y(t_0)$  is a random variable.

$$E[Y(t)] = 0.$$

$$R_Y(t_1, t_2) = E[Y(t_1)Y^*(t_2)] = K_Y(t_1, t_2).$$

$$\begin{aligned} K_Y(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} K_X(s_1, s_2) ds_1 ds_2 \\ &= \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(s_1 - s_2) ds_1 ds_2 \\ &= \sigma^2 \int_0^{t_2} u(t_1 - s_2) ds_2 \\ &= \sigma^2 \int_0^{\min(t_1, t_2)} ds_2 \\ &= \sigma^2 \min(t_1, t_2) \end{aligned}$$

which implies a Wiener process. Note that  $Y(t)$  is Gaussian also. Thus, the M.S. integral of white noise is the Wiener process.