18.0 Wiener Filters

18.1 Discrete Case

$$x(n) \longrightarrow \underbrace{\mathbf{H}(\mathbf{f})}_{causal} \longrightarrow \underbrace{\oplus}_{\uparrow v(n)} \longrightarrow y(n) \longrightarrow \mathbf{G}(\mathbf{f}) \longrightarrow \hat{x}(n)$$

Find linear $G(f)$ so that $E\left[|x(n) - \hat{x}(n)|^2\right]$ is minimized.

Assume $x(n)$ is zero mean, $K_x, K_v, h(n)$ are known, and $E[x(n)v(n)] = 0 \quad \forall n, m.$

We will need

$$K_{xy}(n) = E[x(m)y^*(m-n)]$$

$$= E\left[x(m)\left(\sum_{k=0}^{\infty} h^*(k)x^*(m-n-k) + v^*(m-n)\right)\right]$$

$$K_{xy}(n) = \sum_k h^*(k)K_x(n+k)$$

$$\Longrightarrow \quad [S_{XY}(f) = H^*(f)S_X(f)]$$

$$K_y(n) = E[y(m)y^*(m-n)]$$

$$= \sum_k \sum_\ell h(k)h^*(\ell)K_x(n+\ell-k) + K_v(n)$$

$$\Longrightarrow \quad [S_Y(f) = H(f)S_X(f)H^*(f) + S_v(f)]$$

$$x(n) \longrightarrow \mathbf{H}(\mathbf{f}) \longrightarrow \underbrace{\oplus}_{\uparrow v(n)} \longrightarrow y(n) \longrightarrow \mathbf{A}(\mathbf{f}) \longrightarrow \underbrace{w(n)}_{white_{\sigma^2=1}} \longrightarrow \mathbf{B}(\mathbf{f}) \longrightarrow \hat{x}(n)$$

We wish to min $\{\underbrace{E[|x(n) - \hat{x}(n)|^2]}_{\mathbf{MSE}}\}$

$$\mathbf{MSE} = E[|x(n) - \textstyle\sum_{k=-\infty}^{\infty} b(k)w(n-k)|^2]$$

$$= E[|x(n)|^2] - E[x(n)\sum_k b^*(k)w^*(n-k)]$$

$$-E\left[\left(\sum_k b(k)w(n-k)\right)x^*(n)\right] + E\left[\sum_k\sum_\ell b(k)b^*(\ell)w(n-k)w^*(n-\ell)\right]$$

$$= K_X(0) - 2\sum_k Re\{b^*(k)K_{XW}(k)\} + \underbrace{\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} b(k)b^*\ell K_W(\ell-k)}_{=\sigma_w^2\sum_k |b(k)|^2}$$

$$= K_X(0) + \sum_{k=-\infty}^{\infty}(|b(k)|^2 - 2Re\{b^*(k)K_{XW}(k)\})$$

$$= K_X(0) + \underbrace{\sum_k |b(k) - K_{XW}(k)|^2}_{=0 \quad if \quad b(k)=K_{XW}(k)} - \sum_k |K_{XW}(k)|^2$$

$$w(n) = a(n) * y(n)$$

$$\implies \quad S_{XW}(f) = S_{XY}(f)A^*(f)$$

$$\implies \quad B(f) = S_{XY}(f)A^*(f)$$

$$y(n) \longrightarrow \underbrace{\mathbf{A(f)} \longrightarrow \mathbf{B(f)}}_{\mathbf{G(f)}} \longrightarrow \hat{x}(n)$$

$$G(f) = A(f)B(f)$$

$$= A(f)S_{XY}(f)A^*(f)$$

$$= S_{XY}(f)|A(f)|^2$$

Now, $y(n) \longrightarrow \mathbf{A(f)} \longrightarrow w(n)$

$$S_W(f) = S_Y(f)|A(f)|^2$$

$$\implies |A(f)|^2 = \frac{1}{S_Y(f)}$$

So,

$$\underbrace{G(f) = \frac{S_{XY}(f)}{S_Y(f)}}_{\textbf{Wiener Filter}}$$

Or,

$$\underbrace{G(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{H^*(f)S_X(f)}{H(f)S_X(f)H^*(f) + S_V(f)}}_{\textbf{Wiener Filter}}$$

Recall, LMMSE:

$$\overset{zero\ mean}{\underline{x}} \longrightarrow \mathbf{H} \longrightarrow \underbrace{\oplus}_{\uparrow \underline{v}} \longrightarrow \underline{z}, \qquad \hat{\underline{x}} = G\underline{z}$$

$$G_{matrix} = K_X H^{*T}[HK_X H^{*T} + K_V]^{-1}$$

(note similarity with $G(f)$)

$$x(n) \longrightarrow \mathbf{H(f)} \longrightarrow \underbrace{\oplus}_{\uparrow v(n), E[v(n)]=0} \longrightarrow y(n) \longrightarrow \mathbf{G(f)} \longrightarrow \hat{x}(n)$$

Suppose $x(n)$ is real,

$$\text{MSE} = E\left[(x(n) - \hat{x}(n))^2\right] = E\left[\left(x(n) - \sum_k g(k)y(n-k)\right)^2\right]$$

$$= E\left[x(n)^2\right] - 2E\left[x(n)\sum_k g(k)y(n-k)\right] + E\left[\left(\sum_k g(k)y(n-k)\right)^2\right]$$

MSE $= K_X(0) - 2\sum_k g(k)K_{XY}(k) + \sum_k\sum_\ell g(k)g(\ell)K_Y(k-\ell)$

$$\frac{\partial}{\partial g(m)} = 0 = -2K_{XY}(m) + \underbrace{2\sum_\ell g(\ell)K_Y(m-\ell)}_{by\ symmetry\ of\ K_Y\ for\ real\ sequences}$$

So,

$$K_{XY}(m) = \sum_{\ell=-\infty}^{\infty} g(\ell) K_Y(m-\ell) \qquad -\infty < m < \infty$$

$$\implies \quad S_{XY}(f) = G(f) S_Y(f)$$

Or,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{H^*(f) S_X(f)}{H(f) S_X(f) H^*(f) + S_V(f)}$$

Mean Square Error

Orthogonal Principle $\implies$ error is orthogonal to any function of the data.

$$E[\hat{x}^*(n)(x(m) - \hat{x}(m))] = 0$$

$$\implies \quad E[\hat{x}(n)(x^*(m) - \hat{x}^*(m))] = 0 \quad \forall n, m$$

$$\text{MSE} = E[(x(n) - \hat{x}(n))(x*(n) - \hat{x}^*(n))]$$

$$= E[|x(n)|^2] - E[x(n)\hat{x}^*(n)]$$

$$= K_X(0) - K_{X\hat{X}}(0)$$

In frequency domain,

$$K_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) df$$

$$K_{X\hat{X}}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{X\hat{X}}(f) df$$

$$K_X(k) \overset{F.T.}{\longleftrightarrow} S_X(f)$$

$$K_{X\hat{X}}(k) \overset{F.T.}{\longleftrightarrow} S_{X\hat{X}}(f)$$

$$K_{X\hat{X}}(k) = E[x(n)\hat{x}^*(n-k)]$$

$$= E[x(n) \sum_m g^*(m) y^*(n-k-m)]$$

$$= \sum_m g^*(m) K_{XY}(k+m)$$

$$\implies \quad S_{X\hat{X}}(f) = G^*(f) S_{XY}(f)$$

$$= \left(\frac{S_{XY}(f)}{S_Y(f)}\right)^* S_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_Y(f)}$$

So,

$$\text{MSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[S_X(f) - \frac{|S_{XY}(f)|^2}{S_Y(f)}\right] df$$

Example

$$H(f) = \begin{cases} 1, |f| \leq 0, \\ 0, f_0 < |f| \leq \dfrac{1}{2} \end{cases}$$

$$S_X(f) = e^{-|f|}, \qquad S_V(f) = \sigma_v^2$$

Determine optimum filter G (in mean square sense) to recover x given y.
Know,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)}$$

$$S_{XY}(f) = S_X(f)H^*(f)$$

So,

$$S_{XY}(f) = \begin{cases} e^{-|f|}, |f| \leq 0, \\ 0, f_0 < |f| \leq \dfrac{1}{2} \end{cases}$$

$$S_Y(f) = |H(f)|^2 S_X(f) + S_V(f)$$

$$S_Y(f) = \begin{cases} e^{-|f|} + \sigma_v^2, |f| \leq 0, \\ \sigma_v^2, f_0 < |f| \leq \dfrac{1}{2} \end{cases}$$

So,

$$G(f) = \begin{cases} \dfrac{e^{-|f|}}{e^{-|f|} + \sigma_v^2}, |f| \leq 0, \\ 0, f_0 < |f| \leq \dfrac{1}{2} \end{cases}$$

The Wiener Filter is non-causal!

Causal Constraint

$$g(n) = 0 \quad \forall n < 0$$

Get MSE $= K_X(0) - \sum_{k=0}^{\infty} K_{XW}^2(k)$.

18.2 Continuous Case

A similar analysis as above yields,

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)}$$

$$\text{MSE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[S_X(\omega) - \frac{|S_{XY}(\omega)|^2}{S_Y(\omega)} \right] d\omega$$