

16.0 Stochastic Inputs to LSI Systems

16.1 General Systems

$$y(u, n) = \sum_{m=0}^{\infty} h(m, n)x(u, m)$$

$h(m, n)$ is not shift-invariant in general and $y(u, n)$ does not exist in general (there may be some sample processes that are not summable).

$y(u, n)$ exists in the mean square sense if

- 1) BIBO
- 2) Variances are bounded

Let $y_k(u, n) = \sum_{m=0}^k h(m, n)x(u, m)$. If for every $\epsilon > 0 \exists$ a number $N(\epsilon)$ s.t.

$$E[|y_k(u, n) - y_\ell(u, n)|^2] < \epsilon \quad \forall k, \ell > N(\epsilon)$$

then

$$y_k(u, n) \xrightarrow{m.s.} y(u, n)$$

Consider WLOG that $k > \ell$,

$$\begin{aligned} E[|y_k(n) - y_\ell(n)|^2] &= E\left[\left|\sum_{m=\ell+1}^k h(m, n)x(m)\right|^2\right] \\ &= \sum_{m=\ell+1}^k \sum_{p=\ell+1}^k h(m, n)h^*(n, p)R_X(n, p) \\ &\leq \sum_{m=\ell+1}^k \sum_{p=\ell+1}^k |h(m, n)||h^*(n, p)||R_X(n, p)| \\ &\leq \left[\sum_{p=\ell+1}^k |h(n, p)|(R_X(p, p))^{\frac{1}{2}}\right]^2 \end{aligned}$$

If

$$\lim_{k \rightarrow \infty} \sum_{p=0}^k |h(n, p)|(R_X(p, p))^{\frac{1}{2}}$$

exists, then as $k, \ell \rightarrow \infty$

$$\sum_{p=\ell+1}^k |h(n, p)|(R_X(p, p))^{\frac{1}{2}} \rightarrow 0$$

Theorem: If $X(u, n)$ is a sequence of random variables with $R_X(n, n) < \infty \quad \forall n$ and $h(n, m)$ is absolutely summable, i.e.,

$$\sum_{m=0}^{\infty} |h(m, n)| < \infty \quad \forall n$$

then,

$$\sum_{m=0}^{\infty} |h(n, m)|x(u, m)$$

exists in the mean square sense.

16.2 WSS in LSI Systems

Assume $h(n)$ is causal, LSI, BIBO stable. Recall we have BIBO stability and causality *if and only if* $\sum_{n=0}^{\infty} |h(n)| < \infty$, and $h(n) = 0$ for $n < 0$. Assume $x(n)$ is WSS and $E[x(n)^2] < \infty$.

Mean

$$\begin{aligned} E[y(n)] &= E[h(n) * x(n)] \\ &= \sum_{k=0}^{\infty} h(k)E[x(n-k)] \\ \mu_y &= \mu_x \sum_{k=0}^{\infty} h(k) \quad (\text{constant}) \end{aligned}$$

Cross Correlation

$$\begin{aligned} R_{XY}(n, m) &= E[x(u)y^*(m)] \\ &= E[x(u) \sum_{k=0}^{\infty} h^*(k)x^*(m-k)] \\ &= \sum_{k=0}^{\infty} h^*(k)R_X(n, m-k) \\ &= \sum_k h^*(k)R_X(n-m+k) \end{aligned}$$

Let $\ell = n - m$

$$\begin{aligned} R_{XY}(\ell) &= \sum_{k=0}^{\infty} h^*(k)R_X(\ell + k) \\ &= R_X(\ell) * h^*(-\ell) \end{aligned}$$

Similar derivation for cross covariance, $K_{XY}(n, m)$

Correlation of Y

$$\begin{aligned} R_Y(n, m) &= E[y(n)y^*(m)] \\ &= E\left[\sum_p h(p)x(n-p)y^*(m)\right] \\ &= \sum_{p=0}^{\infty} h(p) \underbrace{E[x(n-p)y^*(m)]}_{R_{XY}(n-p-m)} \\ &= \sum_{p=0}^{\infty} h(p) \sum_{k=0}^{\infty} h^*(k)R_X(n-p-m+k) \end{aligned}$$

Let $q = n - m$

$$\begin{aligned} R_Y(q) &= \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} h(p)h^*(k)R_X(q-p+k) \\ &= h(q) * R_X(q) * h^*(-q) \end{aligned}$$

Recall

$$R_Y = HR_XH^\dagger \quad (\text{note similarity between time domain and freq. domain}).$$

Similar derivation for covariance of Y, $K_Y(n, m)$.

Fourier Space

Define a cross-spectral density as

$$\begin{aligned} S_{XY}(f) &= \sum_{n=-\infty}^{\infty} R_{XY}(n)e^{-i2\pi fn}, f \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} h^*(k)R_X(n+k)e^{-i2\pi f(n+k)}e^{i2\pi fk} \end{aligned}$$

or

$$S_{XY}(f) = S_X(f)H^*(f).$$

Now,

$$\begin{aligned} S_Y(f) &= \sum_{n=-\infty}^{\infty} R_Y(n)e^{-i2\pi fn} \\ &= H(f)H^*(f)S_X(f). \end{aligned}$$

$$S_Y(f) = |H(f)|^2 S_X(f).$$

Recall, $R_Y(q) = h(q) * h^*(-q) * R_X(q)$

Ex1 Suppose $x(n)$ is an i.i.d., zero-mean sequence, $\text{var} = \sigma_X^2$.

$$K_X(n) = \sigma_X^2 \delta(n)$$

$$\begin{aligned} S_X(f) &= \sum_{n=-\infty}^{\infty} K_X(n)e^{-i2\pi fn} \\ &= \sigma_X^2 \quad \longrightarrow \quad \text{”White Noise”} \end{aligned}$$

(power equally spread over all frequencies)

Consider the moving average

$$y(n) = \sum_{k=0}^M b_k x(n - k).$$

Then

$$H(f) = \sum_{k=0}^M b_k e^{-i2\pi f k}.$$

Thus

$$S_Y(f) = \left| \sum_{k=0}^M b_k e^{-i2\pi f k} \right|^2 \sigma_X^2.$$

Now consider the special case $M = 1$, $b_0 = h(0) = 1$, $b_1 = h(1) = -1$ and $h(n) = 0$ for $n \neq 0, 1$. Then

$$y(n) = x(n) - x(n - 1).$$

So

$$H(f) = \sum_{k=0}^1 h(k) e^{-i2\pi f k} = 1 - e^{-i2\pi f}.$$

$$|H(f)|^2 = 2 - 2 \cos(2\pi f)$$

and

$$S_Y(f) = \sigma_X^2 (2 - 2 \cos(2\pi f)).$$