14.0 LTI Systems

14.1 Definitions

Here let H be a mapping

$$H : L_1 \to L_2$$

where, L_1 and L_2 are two linear spaces.

Definition: H is said to be a linear system if

$$H(ax) = aH(x)$$

$$H(x_1 + x_2) = H(x_1) + H(x_2)$$

for scalar a and $x_1, x_2 \in L_1$.

Translation (Shift) Operations

Let T be an Abelian (commutative) group with binary operation "+", i.e.,

1. $t_1, t_2 \in T \Rightarrow t_1 + t_2 \in T$
2. $t_1, t_2, t_3 \in T \Rightarrow t_1 + (t_2 + t_3) = (t_1 + t_2) + t_3)$
3. $\exists\ 0 \in T$ such that $t + 0 = 0 + t = t\ \forall\ t \in T$
4. For every $t \in T\ \exists\ t^{-1} \in T$ such that $t + t^{-1} = 0$ $[t^{-1} = -t]$
5. (Abelian) For every $t_1, t_2 \in T,\ t_1 + t_2 = t_2 + t_1$

Definition: The shift operator T_τ is defined as

$$T_\tau(x)(t) = x(t + \tau)$$

for $t \in T$, "+" is the binary operation. Here T is an Abelian group.

Note: $T_{\tau_1}(T_{\tau_2(x)}) = T_{\tau_1+\tau_2}(x)$.

Definition: A linear system H is said to be *time invariant* or *shift invariant* if

$$H(T_\tau(x)) = T_\tau(H(x))$$

i.e., H commutes with T_τ.

Eigenfunctions
If H is an LTI (or LSI) system then the functions

$$e_f(t) = e^{i2\pi ft}, \ \forall \ t \in T$$

are the system *eigenfunctions*, i.e.,

$$He_f = H(f)e_f$$

where,

i. $f \in \{0, \ 1/n, \ 2/n, \dots, \ (n-1)/n\}$ for $T = [0, 1, 2, \dots, n-1]$

ii. $f \in [-1/2, \ 1/2]$ for $T = \mathbf{Z}$

iii. $f \in [0, \ \pm 1/T, \ \pm 2/T, \dots]$ for $T = [0, A)$

iv. $f \in [-\infty, \ \infty]$ for $T = \mathbf{R}$

14.2 Discrete Time Systems

14.2.1 Eigensequences

Let H be a discrete time invariant linear system with $T = \mathbf{Z}$, so $t \in \{0, \pm 1, \pm 2, \dots\}$. Let

$$e_f(n) = e^{i2\pi fn}.$$

We will show

$$He_f = H(f)e_f$$

where

$$H(f) = \sum_k h(k)e^{-i2\pi fk}.$$

Now

$$He_f = H(e_f)(n) = h(n) * e_f(n)$$
$$= \sum_k h(k)e^{i2\pi f(n-k)} = e^{i2\pi fn} \sum_k h(k)e^{-i2\pi fk} = e^{i2\pi fn}H(f) = H(f)e_f.$$

Now consider $x(n) = e^{i2\pi fn}$. If $x(n)$ is operated on by H then the output $y(n)$ is

$$y(n) = e^{i2\pi fn}H(f)$$

where $H(f)$ is a constant for a fixed f. Now let D^{-k} denote a delay of the input by k samples. Then

$$D^{-k}H\{x(n)\} = HD^{-k}\{x(n)\}.$$

Define the impulse response

$$h(n) = H\{\delta(n)\}$$

where $\delta(n)$ is the delta function that has the value 1 at $n = 0$ and is 0 otherwise. Then

$$x(n) = \sum_k x(k)\delta(n-k)$$

which is a weighted sum of impulses. Thus,

$$h(n) = H\{x(n)\} = H\left\{\sum_k x(k)\delta(n-k)\right\} = \sum_k x(k)H\{\delta(n-k)\}.$$

Now

$$H\{\delta(n-k)\} = HD^{-k}\{\delta(n)\} = D^{-k}H\{\delta(n)\} = D^{-k}h(n) = h(n-k).$$

So

$$y(n) = \sum_k x(k)h(n-k) = \sum_k h(k)x(n-k).$$

Now let $x(n)$ be the eigensequence $e^{i2\pi fn}$. Then

$$y(n) = H(f)x(n) = x(n)H(f) = e^{i2\pi fn}\sum_k h(k)e^{-i2\pi fn}.$$

Note

$$H(f) = \sum_k h(k)e^{-i2\pi fn}$$

is an eigenvalue for a fixed f. $H(f)$ is a Fourier series and denotes the frequency response. Thus, $h(k)$ are the Fourier series coefficients of $H(f)$, i.e.,

$$h(k) = \int_{-1/2}^{1/2} H(f)e^{i2\pi fk}df.$$

Note that $H(f)$ is periodic with period $1 = 1/2 - (-1/2)$. Also, $H(f)$ exists for systems for which $\sum_k |h(k)| < \infty$.

14.2.2 Fourier Analysis

Note that if $y(n) = x(n) * h(n)$ then $Y(f) = X(f)H(f)$.

Now consider the case

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k).$$

Then

$$Y(f) = X(f) \sum_{k=0}^{M} b_k e^{-i2\pi fk} - Y(f) \sum_{i=1}^{N} a_k e^{-i2\pi fk}.$$

So

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^{M} b_k e^{-i2\pi fk}}{1 + \sum_{i=1}^{N} a_k e^{-i2\pi fk}}$$

which is the frequency response of the system.