13.0 Random Processes

13.1 Introduction

Definition: A random process is a set of indexed random variables $X(u,t)$ defined on (U,T,P) where t takes values in some index set T.

For any fixed $t = t_0 \in T, X(u,t_0)$ is a random variable.
For a fixed $u = u_0 \in U, X(u_0,t)$ is a sample function.

If T is finite, we have a random vector.
If T is countable, we have a random sequence.
If $T = \mathbf{R}$, we have a random process.
If $T = \mathbf{R}^n$, we have a random field.
The case $T = \mathbf{R}^2$ is used in image processing.

We often write $X(t)$ for $X(u,t)$

Characterization of Random Process:
Random Variable: $F_X(x) = P(X \leq x)$
First order distribution and density of a random process,

$$F_X(u,t) = P(X(u,t) \leq x),$$

$$f_X(u,t) = \frac{dF_X(u,t)}{dx}.$$

In general, random variables for different $t \in T$ are neither independently nor identically distributed, so 1^{st} order pdf does not characterize the random process.

N^{th} order distribution and pdf:

$$F_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = P(X(u,t_1) \leq x_1 \cdots, X(u,t_n) \leq x_n)$$

leads to $F_X(x_1, \cdots, x_n; t_1, \cdots, t_n)$ which contains all information available. This is usually too complicated to work with. Instead we rely on 1^{st} and 2^{nd} order statistics. Note that these completely characterize the Gaussian case and is often good enough for other distributions.

13.2 The Second Moment Theory of Random Processes

Mean:

$$\mu_X(t) = E[X(u,t)] \qquad \forall\ t \in T$$
$$= \int_{-\infty}^{\infty} x f_X(x,t)\,dx$$

Correlation:

$$R_X(t_1,t_2) = E[X(u,t_1)X^*(u,t_2)] \quad \forall\, t_1,t_2 \in T.$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

Covariance:

$$K_X(t_1,t_2) = E[(X(u,t_1) - \mu_X(t_1))(X(u,t_2) - \mu_X(t_2))^*]$$
$$K_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

13.3 Examples of Random Processes

1)

$$X(u,t) = A(u) \quad \forall \mu \in U, t \in T$$

$A(u)$ is a random variable with mean m and variance σ^2

$$\mu_X(t) = E[A(u)] = m \quad \textit{(not dependent on t)}$$
$$R_X(t_1,t_2) = E[X(u,t_1)X^*(u,t_2)] = E[A(u)A(u)] = \sigma^2 + m^2$$
$$K_X(t_1,t_2) = \sigma^2$$

Note When we compute $\mu_X(t)$ as $E[X(u,t)]$, then we are in effect computing the ensemble average for each t. Similarly for R_X and K_X.

Say, $Z(u) \sim N(0, \sigma^2)$. $X(u,t) = Z(u)$

$$\mu_X(t) = 0, K_X(t_1, t_2) = \sigma^2$$

Let us observe this $X(u,t)$ over time t. $X(u,t)$ does not change over time. We just observe some constant sample and if we do this many times on the average the constant will be 0 but any particular outcome, i.e., $X(u_0, t)$ is some constant $Z(u_0)$.

So, time average of $X(u_0, t)$ is a constant and not necessarily equal to the ensemble average for some $X(u, t_0)$.

If time average equals to ensemble average, we have an ergodic process.

2)

$$X(u,t) = \sin(t - \phi(u))$$
$$\phi(u) \sim U(-\pi, \pi)$$

So,

$$f_\phi(\phi) = \begin{cases} \dfrac{1}{2\pi} & |\phi| < \pi, \\ \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \mu_X(t) &= E[\sin(t - \phi(u))] \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin(t - \phi) d\phi \\ &= \frac{1}{2\pi} \cos(t - \phi)|_{-\pi}^{\pi} = 0 \end{aligned}$$

$$K_X(t_1, t_2) = R_X(t_1, t_2)$$
$$E[\sin(t_1 - \phi)\sin(t_2 - \phi)]$$
$$= \frac{1}{2} E[\cos(t_1 - t_2) - \cos(t_1 + t_2 - 2\phi)]$$
$$= \frac{1}{2}\cos(t_1 - t_2) - \frac{1}{4\pi} \underbrace{\int_{-\pi}^{\pi} \cos(t_1 + t_2 - 2\phi) d\phi}_{0}$$

$$K_X(t_1, t_2) = \frac{1}{2}\cos(t_1 - t_2)$$

<u>Note</u> $K_X(t_1, t_2)$ is a function of $(t_1 - t_2)$ only. So,

$$K_X(t_1 + \tau, t_2 + \tau) = \frac{1}{2}\cos(t_1 - t_2)$$

This is 2^{nd} order stationarity.

13.4 Properties of Correlation Functions

1) $\mu_X(t)$ is any real function defined on T.
2) $R_X(t_1, t_2) = R_X^*(t_2, t_1)$
3) $R_X(t_1, t_2)$ is a non-negative definite function.
4) $R_X(t, t) \geq 0 \quad \forall t \in T$.
5) $|R_X(t_1, t_2)| \leq \sqrt{R_X(t_1, t_1)}\sqrt{R_X(t_2, t_2)}$

1),2),3) are necessary and sufficient condition for the existence of a random process with mean $\mu_X(t)$ and correlation $R_X(t_1, t_2)$.