

## 8.0 Projections and K-L Expansions

### 8.1 Projection of Random Vectors

**Definition:** A vector  $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_n)^t$  is called a *directional vector* if  $\|\mathbf{e}\| = 1$ . Here

$$\|\mathbf{e}\|^2 = \sum_{i=1}^n |\mathbf{e}_i|^2$$

**Definition:** The projection of a random vector  $\mathbf{X}(u)$  in a direction  $\mathbf{e}$  is

$$\mathbf{X}_e(u) = \langle \mathbf{X}(u), \mathbf{e} \rangle = \mathbf{e}^\dagger \mathbf{X}(u)$$

If  $\mu_X = 0$ , get

$$\sigma_{X_e}^2 = \mathbf{e}^\dagger \mathbf{K}_X \mathbf{e}$$

Suppose  $\mathbf{e}$  is a normalized eigenvector of  $\mathbf{K}_X$  with eigenvalue  $\lambda$ , then

$$\sigma_{X_e}^2 = \lambda \|\mathbf{e}\|^2 = \lambda$$

Now let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be an orthonormal basis of an n-dimensional vector space where each  $\mathbf{e}_k$  is an eigenvector of  $\mathbf{K}_X$  with eigenvalue  $\lambda_k$ . Then, with  $\mathbf{e}$  as a directional vector

$$\mathbf{e} = \sum_{k=1}^n a_k \mathbf{e}_k, \quad \text{some } a_k$$

or,  $\mathbf{e} = \mathbf{E}\mathbf{a} \implies \mathbf{a} = \mathbf{E}^{-1}\mathbf{e}$ , where  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n)^t$

or,

$$\mathbf{a}_k = \mathbf{e}_k^\dagger \mathbf{e}$$

$\mathbf{a}_k$  is the projection of  $\mathbf{e}$  onto the  $k^{\text{th}}$  eigenvector.

We have

$$1 = \|\mathbf{e}\|^2 = \|\mathbf{E}\mathbf{a}\|^2 = (\mathbf{E}\mathbf{a})^\dagger \mathbf{E}\mathbf{a} = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{E}\mathbf{a} = \mathbf{a}^\dagger \mathbf{a} = \|\mathbf{a}\|^2$$

So the transformation is norm-preserving.

With  $\mathbf{e} = \sum_{k=1}^n \mathbf{a}_k \mathbf{e}_k$ , we get

$$\sigma_{X_e}^2 = \mathbf{e}^\dagger \mathbf{K}_X \mathbf{e}$$

$$\begin{aligned}
&= \left( \sum_j \mathbf{a}_j \mathbf{e}_j \right)^\dagger \mathbf{K}_X \left( \sum_k \mathbf{a}_k \mathbf{e}_k \right) \\
&= \sum_j \sum_k \mathbf{a}_j^* \mathbf{a}_k \mathbf{e}_j^\dagger \mathbf{K}_X \mathbf{e}_k \\
&= \sum_j \sum_k \mathbf{a}_j^* \mathbf{a}_k \mathbf{e}_j^\dagger \lambda_k \mathbf{e}_k \\
&= \sum_j \sum_k \lambda_k \mathbf{a}_j^* \mathbf{a}_k \langle \mathbf{e}_k, \mathbf{e}_j \rangle \\
&= \sum_j \sum_k \lambda_k \mathbf{a}_j^* \mathbf{a}_k \delta(k - j) \\
&= \sum_k \lambda_k |\mathbf{a}_k|^2
\end{aligned}$$

But,  $\sum_k |\mathbf{a}_k|^2 = 1$ , so (think why?)

$$\lambda_{min} \leq \sigma_{X_e}^2 \leq \lambda_{max}$$

or,  $\mathbf{e} = \mathbf{E}\mathbf{a} \implies \mathbf{e}^\dagger \mathbf{X} = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{X}$

$$\begin{aligned}
\sigma_{X_e}^2 &= E[\mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{X} \mathbf{X}^\dagger \mathbf{E} \mathbf{a}] = \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{K}_X \mathbf{E} \mathbf{a} \\
&= \mathbf{a}^\dagger \mathbf{E}^\dagger \mathbf{E} \boldsymbol{\Lambda} \mathbf{a} = \mathbf{a}^\dagger \boldsymbol{\Lambda} \mathbf{a} = \sum_k \lambda_k |\mathbf{a}_k|^2 \\
&\implies \lambda_{min} \leq \sigma_{X_e}^2 \leq \lambda_{max}
\end{aligned}$$

## 8.2 Karhunen-Loeve Expansion

$$\begin{aligned}
 E[|\mathbf{X}|^k] &= \int |x|^k f_X(x) dx \geq \int_{|x| \geq \alpha} |x|^k f_X(x) dx \\
 &\geq \alpha^k \int_{|x| \geq \alpha} f_X(x) dx \quad (\alpha > 0) \\
 &= \alpha^k P(|X| \geq \alpha)
 \end{aligned}$$

So,  $P(|X| \geq \alpha) \leq \frac{1}{\alpha^k} E[|X|^k]$ . This is Markov's Inequality.

If we set  $k=2$  and subtract  $\mu = E(X)$  from  $X$ , we get Chebyshev's Inequality:

$$P(|X - \mu| > \alpha) \leq \frac{1}{\alpha^2} \underbrace{E(|X - \mu|^2)}_{\text{Var}(X)}$$

**Theorem:** Let  $\mathbf{X}(u)$  be a zero-mean random vector and let  $\tilde{\mathbf{X}}_k(u) = \langle \mathbf{X}(u), \mathbf{e}_k \rangle$  where the  $\mathbf{e}_k$  is an orthonormal basis of eigenvectors of  $\mathbf{K}_X$ . Then,

$$\mathbf{X}(u) = \sum_{k=1}^n \tilde{\mathbf{X}}_k(u) \mathbf{e}_k \quad (K-L \text{ expansion in probability})$$

Furthermore,

$$E[\tilde{\mathbf{X}}_k(u)] = 0$$

and,

$$E[\tilde{\mathbf{X}}_k(u) \tilde{\mathbf{X}}_m^*(u)] = \lambda_m \delta(m - k)$$

**Proof:** Clearly  $E[\tilde{\mathbf{X}}_k(u)] = 0$ .

$$\begin{aligned}
 E[\tilde{\mathbf{X}}_k(u) \tilde{\mathbf{X}}_m^*(u)] &= E[\langle \mathbf{X}(u), \mathbf{e}_k \rangle \langle \mathbf{X}(u), \mathbf{e}_m \rangle^*] \\
 &= E[(\mathbf{e}_k^\dagger \mathbf{X}(u)) (\mathbf{e}_m^\dagger \mathbf{X}(u))^*] \\
 &= E[\mathbf{e}_k^\dagger \mathbf{X}(u) \mathbf{X}^\dagger(u) \mathbf{e}_m] \\
 &= \mathbf{e}_k^\dagger \mathbf{K}_x \mathbf{e}_m \\
 &= \mathbf{e}_k^\dagger \lambda_m \mathbf{e}_m
 \end{aligned}$$

$$= \lambda_m \delta(m - k)$$

To say  $\mathbf{X}(u) = \sum_{k=1}^n \tilde{\mathbf{X}}_k(u) \mathbf{e}_k$  in probability we mean for any  $\epsilon > 0$

$$P[\|\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\| \geq \epsilon] = 0 \quad (\text{for any } \epsilon > 0)$$

Now, by Chebyshev's Inequality,

$$P[\|\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\| \geq \epsilon] \leq \frac{E\left[\|\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\|^2\right]}{\epsilon^2}$$

$$\begin{aligned} E\|\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\|^2 &= \mathbf{E}\left[\left(\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\right)^\dagger \left(\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\right)\right] \\ &= E\left[\|\mathbf{X}(u)\|^2 - \mathbf{X}^\dagger(u) \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k - \left(\sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\right)^\dagger \mathbf{X}(u) \right. \\ &\quad \left. + \left(\sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\right)^\dagger \left(\sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\right)\right] \end{aligned}$$

Since

$$\tilde{\mathbf{X}}_k(u) = \mathbf{e}_k^\dagger \mathbf{X}(u)$$

We have that

$$\begin{aligned} \mathbf{X}^\dagger(u) \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k &= \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{X}^\dagger(u) \mathbf{e}_k \\ &= \sum_k \tilde{\mathbf{X}}_k(u) (\mathbf{e}_k^\dagger \mathbf{X}(u))^\dagger = \sum_k \tilde{\mathbf{X}}_k \tilde{\mathbf{X}}_k^* \\ &= \sum_k |\tilde{\mathbf{X}}_k|^2 \end{aligned}$$

Therefore,

$$\begin{aligned} E\|\mathbf{X}(u) - \sum_k \tilde{\mathbf{X}}_k(u) \mathbf{e}_k\|^2 &= E[\|\mathbf{X}(u)\|^2] - 2 \sum_k E|\tilde{\mathbf{X}}_k(u)|^2 + \sum_k \sum_m E[\tilde{\mathbf{X}}_k^*(u) \tilde{\mathbf{X}}_m(u)] \delta(m-k) \\ &= \text{Tr}(\mathbf{K}_X) - 2 \sum_k E|\tilde{\mathbf{X}}_k(u)|^2 + \sum_k E|\tilde{\mathbf{X}}_k(u)|^2 \end{aligned}$$

$$\begin{aligned} &= \text{Tr}(\mathbf{K}_X) - \sum_k E|\tilde{\mathbf{X}}_k(u)|^2 \\ &= \sum_{k=1}^n \lambda_k - \sum_{k=1}^n \lambda_k = 0 \end{aligned}$$