6.0 Causal Solution to Spectral Shaping

Consider

$$\mathbf{X} = \mathbf{H}\mathbf{W}.$$

Definition: A transformation $\mathbf{H}$ is *causal* if $\mathbf{H}$ is a lower triangular square matrix:

$$H = \begin{bmatrix} h_{11} & 0 & \dots & 0 \\ h_{21} & h_{22} & \dots & 0 \\ \vdots & \dots & & \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{bmatrix}.$$

Here

$$X_i(u) = X(u,i) = \sum_{j=1}^{n} h_{ij} W_j(u)$$

or

$$X_i(u) = \sum_{j=1}^{i} h_{ij} W_j(u) \text{ for } \mathbf{H} \text{ causal.}$$

If $\mathbf{K_W} = \mathbf{I}$ then

$$\mathbf{K_X} = \mathbf{H}\mathbf{H}^\dagger.$$

We write

$$\mathbf{K_X} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \vdots & \dots & & \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix}.$$

6.1 Direct Method for Causal Factorization

$$k_{11} = h_{11}h_{11}^* = |h_{11}|^2, \quad k_{12} = h_{11}h_{21}^*, \ etc.$$

We can solve for the h_{ij} this way.

6.2 Row Operation Method for Causal Factorization

The row operation method involves performing row operations to write (for $\mathbf{K}$ full rank)

$$\mathbf{L}\mathbf{K} = \mathbf{U}$$

where $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular. $\mathbf{L}$ is the product of lower triangular matrices that perform row operations on $\mathbf{K}$. So

$$\mathbf{K} = \mathbf{L}^{-1}\mathbf{U}.$$

$\mathbf{L}^{-1}$ is also lower triangular. Now we can write $\mathbf{K}$ as

$$\mathbf{K} = \tilde{\mathbf{L}}\mathbf{D}\tilde{\mathbf{L}}^{\dagger}$$

where $\tilde{\mathbf{L}} = \mathbf{L}^{-1}$ and $\mathbf{D}\tilde{\mathbf{L}}^{\dagger} = \mathbf{U}$. $\mathbf{D}$ is the diagonal of $\mathbf{U}$. So

$$\mathbf{U} = \mathbf{D}\left(\mathbf{L}^{-1}\right)^{\dagger} \Rightarrow \mathbf{K} = \mathbf{L}^{-1}\mathbf{D}\left(\mathbf{L}^{-1}\right)^{\dagger}.$$

Let

$$\mathbf{H} = \mathbf{L}^{-1}\mathbf{D}^{1/2}.$$

This is okay since $d_{ij} \geq 0$. Note

$$\mathbf{L}^{-1}\mathbf{D}^{1/2} = \left(\mathbf{D}^{-1}\mathbf{U}\right)^{\dagger}\mathbf{D}^{1/2} = \mathbf{U}^{\dagger}\mathbf{D}^{-1/2} = \left(\mathbf{D}^{-1/2}\mathbf{U}\right)^{\dagger}.$$

So

$$\mathbf{H} = \left(\mathbf{D}^{-1/2}\mathbf{U}\right)^{\dagger}$$

is a causal solution.

Observe we do not actually need to find $\mathbf{D}$ to form $\mathbf{H}$.

To form $\mathbf{H}^{\dagger}$

1. Perform row operations on $\mathbf{K}$ until upper triangular $\mathbf{U}$ is found.
2. Divide each row of $\mathbf{U}$ by the square root of the element on the corresponding main diagonal in the row.

If $\mathbf{K}$ is not full rank then we will get at least one zero row when forming $\mathbf{U}$. In this case simply leave those rows untouched when dividing by the square root of the diagonal elements.

Cholesky Decomposition

__**Theorem:**__ Let $\mathbf{K}$ be a $n \times n$ positive definite Hermitian symmetric matrix. Then we can write

$$\mathbf{K} = \mathbf{L}\mathbf{L}^\dagger$$

where $\mathbf{L}$ is lower triangular with positive nonzero entries on the diagonal.

Proof: By induction. For $n = 1$, $\mathbf{K} = (k_{11}),\ k_{11} > 0$ so $\mathbf{L} = \mathbf{L}^\dagger = \sqrt{k_{11}}$. Suppose true for $n-1$. Partition $\mathbf{K}$ as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{n-1,n-1} & \mathbf{b} \\ \mathbf{b}^\dagger & k_{nn} \end{bmatrix}.$$

Since $\mathbf{K}_{n-1,n-1}$ is a principal submatrix of a positive definite matrix it is itself positive definite, $k_{nn} > 0$, real and $\mathbf{b}$ is $(n-1) \times 1$. By induction

$$\mathbf{K}_{n-1,n-1} = \mathbf{L}_{n-1,n-1}\mathbf{L}^\dagger_{n-1,n-1}.$$

We look for $\mathbf{L}$ as

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{n-1,n-1} & \underline{0} \\ \mathbf{c}^\dagger & \alpha \end{bmatrix}.$$

So,

$$\begin{bmatrix} \mathbf{K}_{n-1,n-1} & \mathbf{b} \\ \mathbf{b}^\dagger & k_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{n-1,n-1} & \underline{0} \\ \mathbf{c}^\dagger & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{L}_{n-1,n-1} & \mathbf{c} \\ \underline{0} & \alpha \end{bmatrix}.$$

Thus,

$$\mathbf{L}_{n-1,n-1}\mathbf{c} = \mathbf{b}$$

and

$$\mathbf{c}\mathbf{c}^\dagger + \alpha^2 = k_{nn}.$$

So

$$\mathbf{c} = \mathbf{L}^{-1}_{n-1,n-1}\mathbf{b}$$

$$0 < det(\mathbf{K}) = \alpha^2 \cdot [det(\mathbf{L}_{n-1,n-1})]^2$$

thus α^2 is positive and real. We can solve

$$||\mathbf{c}||^2 + \alpha^2 = k_{nn}$$

for $\alpha > 0$.