5.0 Spectral Shaping Problem

Consider

$$\mathbf{X} = \mathbf{H}\mathbf{W} + \mathbf{a}$$

where $\mathbf{W}$ is mean 0 and $\mathbf{K_W} = \mathbf{I}$. We know

$$\mu_{\mathbf{X}} = \mathbf{a}, \quad \mathbf{K_X} = \mathbf{H}\mathbf{K_W}\mathbf{H}^{\dagger} = \mathbf{H}\mathbf{H}^{\dagger}.$$

Finding $\mathbf{H}$ to give us some desired $\mathbf{K_X}$ is called *spectral shaping.*

The diagonal form of $\mathbf{K_X}$ is

$$\mathbf{K_X} = \mathbf{E_X}\Lambda_{\mathbf{X}}\mathbf{E}_{\mathbf{X}}^{\dagger}.$$

Let

$$\Lambda_{\mathbf{X}}^{1/2} = diag\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\right), \ \lambda_i \geq 0 \ real.$$

Then,

$$\mathbf{K_X} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\Lambda_{\mathbf{X}}^{1/2}\mathbf{E}_{\mathbf{X}}^{\dagger} = \left(\mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\right)\left(\mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\right)^{\dagger}.$$

Let

$$\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}$$

then

$$\mathbf{K_X} = \mathbf{H}\mathbf{H}^{\dagger}$$

as desired. So this $\mathbf{H}$ is a solution to the spectral shaping problem. This solution is not unique. Consider

$$\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}$$

where $\mathbf{U}$ is any unitary matrix. Then,

$$\begin{aligned}
\mathbf{H}\mathbf{H}^{\dagger} &= \left(\mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}\right)\left(\mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}\right)^{\dagger} \\
&= \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}\mathbf{U}^{\dagger}\Lambda_{\mathbf{X}}^{1/2}\mathbf{E}_{\mathbf{X}}^{\dagger} \\
&= \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{I}\Lambda_{\mathbf{X}}^{1/2}\mathbf{E}_{\mathbf{X}}^{\dagger} \\
&= \mathbf{E_X}\Lambda_{\mathbf{X}}\mathbf{E}_{\mathbf{X}}^{\dagger} \\
&= \mathbf{K_X}
\end{aligned}$$

so, $\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}$ is also a solution.

Claim: $\mathbf{H} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}$ gives us all solutions to our spectral shaping problem, where $\mathbf{U}$ is a unitary matrix.

Proof: Let $\tilde{\mathbf{H}}$ be a solution to the spectral shaping problem. Then

$$\mathbf{K_X} = \tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\dagger}.$$

Let

$$\tilde{\mathbf{U}} = \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}$$

so

$$\tilde{\mathbf{H}} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\tilde{\mathbf{U}}.$$

Then,

$$\begin{aligned}
\tilde{\mathbf{U}}\tilde{\mathbf{U}}^{\dagger} &= \left(\Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\right)\left(\Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\right)^{\dagger} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{E}_{\mathbf{X}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}\mathbf{E}_{\mathbf{X}}^{\dagger}\mathbf{E_X}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\mathbf{I}\Lambda_{\mathbf{X}}\mathbf{I}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \Lambda_{\mathbf{X}}^{-1/2}\Lambda_{\mathbf{X}}\Lambda_{\mathbf{X}}^{-1/2} \\
&= \mathbf{I}.
\end{aligned}$$

So, $\tilde{\mathbf{U}}$ is a unitary matrix, call it $\mathbf{U}$. Hence,

$$\tilde{\mathbf{H}} = \mathbf{E_X}\Lambda_{\mathbf{X}}^{1/2}\mathbf{U}.$$