4.0 Analysis of Correlation Matrices

4.1 Decomposition of the Correlation Matrix

Recall $\mathbf{R_X}$ is non-negative definite.

Proposition: If a matrix $\mathbf{M}$ is Hermitian symmetric then

i. all its eigenvalues are real.

ii. the eigenvectors corresponding to distinct eigenvalues are orthogonal.

iii. there exists an orthonormal basis of the n-dimensional vector space which consists of the normalized eigenvectors of $\mathbf{M}$.

iv. $\mathbf{M}$ is non-negative definite if and only if all its eigenvalues are non-negative.

Proof:

i. Let λ be an eigenvalue and $\mathbf{e}$ a normalized eigenvector such that $||\mathbf{e}|| = 1$. Then

$$\mathbf{e}^\dagger \mathbf{M} \mathbf{e} = \mathbf{e}^\dagger \lambda \mathbf{e} = \lambda ||\mathbf{e}||^2.$$

$$\mathbf{e}^\dagger \mathbf{M} \mathbf{e} = \left(\mathbf{e}^\dagger \mathbf{M}\right) \mathbf{e} = \left(\mathbf{M}^\dagger \mathbf{e}\right)^\dagger \mathbf{e} = (\mathbf{M}\mathbf{e})^\dagger \mathbf{e} = (\lambda \mathbf{e})^\dagger \mathbf{e} = \lambda^* ||\mathbf{e}||^2.$$

Thus, $\lambda = \lambda^*$ so λ is real.

ii Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of $\mathbf{M}$ and let $\mathbf{e}_1,\ \mathbf{e}_2$ be the corresponding eigenvectors. Then,

$$\mathbf{e}_1^\dagger \mathbf{M} \mathbf{e}_2 = \mathbf{e}_1^\dagger \lambda_2 \mathbf{e}_2 = \lambda_2 < \mathbf{e}_1, \mathbf{e}_2 > .$$

$$\mathbf{e}_1^\dagger \mathbf{M} \mathbf{e}_2 = (\mathbf{e}_1^\dagger \mathbf{M}) \mathbf{e}_2 = (\mathbf{M} \mathbf{e}_1)^\dagger \mathbf{e}_2 = \lambda_1^* \mathbf{e}_1^\dagger \mathbf{e}_2 = \lambda_1 < \mathbf{e}_1, \mathbf{e}_2 > .$$

Thus

$$(\lambda_2 - \lambda_1) < \mathbf{e}_1^\dagger, \mathbf{e}_2 >= 0 \Rightarrow < \mathbf{e}_1^\dagger, \mathbf{e}_2 >= 0.$$

iii. This follows from ii. when all eigenvalues are real. For the general case see a text on linear algebra.

iv. Show

$$\mathbf{x}^\dagger \mathbf{M}\mathbf{e} \geq 0 \Leftrightarrow \text{ all } \lambda_i \geq 0.$$

" $\Rightarrow$ " only if part or necessity. Assume $\mathbf{x}^\dagger \mathbf{M}\mathbf{e} \geq 0 \;\; \forall$ vectors $\mathbf{x}$. Let $\mathbf{x} = \mathbf{e}$. Then

$$\mathbf{M}\mathbf{e} = \lambda \mathbf{e} \Rightarrow \mathbf{e}^\dagger \mathbf{M}\mathbf{e} = \lambda ||\mathbf{e}||^2.$$

$$\mathbf{e}^\dagger \mathbf{M}\mathbf{e} \geq 0 \Rightarrow \lambda ||\mathbf{e}||^2 \geq 0 \Rightarrow \lambda \geq 0.$$

" $\Leftarrow$ " if part or sufficiency. See a text on linear algebra.

4.2 The Spectral Decomposition of the Correlation Matrix

We have a random vector $\mathbf{X}(u)$ with corresponding correlation matrix $\mathbf{R}_\mathbf{X}$. We know $\mathbf{R}_\mathbf{X}$ is non-negative definite. So there exists an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of eigenvectors of $\mathbf{R}_\mathbf{X}$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then

$$\mathbf{R}_\mathbf{X} \mathbf{e}_i = \lambda_i \mathbf{e}_i, \quad i = 1, 2, \ldots, n.$$

Then

$$\mathbf{R}_\mathbf{X} \left[\mathbf{e}_1 \ldots \mathbf{e}_n\right] = \left[\lambda_1 \mathbf{e}_1 \ldots \lambda_n \mathbf{e}_n\right].$$

Now let

$$\mathbf{E} = \left[\mathbf{e}_1 \ldots \mathbf{e}_n\right], \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

We can then write

$$\mathbf{R}_\mathbf{X} \mathbf{E} = \mathbf{E}\Lambda \Rightarrow \mathbf{R}_\mathbf{X} = \mathbf{E}\Lambda\mathbf{E}^{-1}.$$

Note that $\mathbf{E}$ is a unitary matrix.

Definition: A square matrix $\mathbf{U}$ is said to be a *unitary matrix* if

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}.$$

So $\mathbf{U}^{-1} = \mathbf{U}^\dagger$.

Claim: For $\mathbf{U}$ unitary, every eigenvalue of $\mathbf{U}$ has absolute value $|\lambda| = 1$.

Proof: Let $\mathbf{x}$ be an eigenvector of $\mathbf{U}$. Then

$$\mathbf{U}\mathbf{x} = \lambda \mathbf{x}.$$

But

$$(\mathbf{Ux})^\dagger (\mathbf{Ux}) = \mathbf{x}^\dagger \mathbf{U}^\dagger \mathbf{Ux} = \mathbf{x}^\dagger \mathbf{x} \Rightarrow ||\mathbf{Ux}||^2 = ||\mathbf{x}||^2 \Rightarrow ||\mathbf{Ux}|| = ||\mathbf{x}||.$$

So

$$||\mathbf{Ux}|| = ||\lambda \mathbf{x}|| \Rightarrow ||\mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \Rightarrow |\lambda| = 1.$$

By computing $\mathbf{E}^\dagger \mathbf{E}$ we find that $\mathbf{E}$ is unitary so that $\mathbf{E}^{-1} = \mathbf{E}^\dagger$. Hence

$$\mathbf{R_X} = \mathbf{E} \Lambda \mathbf{E}^\dagger.$$

The above is called the *Spectral Decomposition of* $\mathbf{R_X}$. We may write the last expression in the form

$$\mathbf{R_X} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \mathbf{e}_i^\dagger.$$

The above is called the *Spectral Expansion of* $\mathbf{R_X}$.