# 3.0 Gaussian Random Vectors

An important class of random vectors is the Gaussian (or normal) random vectors.

## 3.1 Gaussian Random Variables

**Definition:** A random variable $X$ is a *Gaussian random variable* if its density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in (-\infty, \infty).$$

We write $X \sim N(\mu, \sigma^2)$.

The distribution function for $X$ is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-\mu)^2/2\sigma^2} dt.$$

Results from the Theory of Equations prove there can be no closed form expression for this integral, i.e., there does not exist a function containing a finite number of terms whose derivative equals $f_X(x)$. So to evaulate this integral we have to make use of tables or series approximations or functional approximations.

**Definition:** The *characteristic function* of a random variable $X$ is given by

$$\Phi_X(\omega) = E\left[e^{i\omega X}\right] = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) dx, \quad i = \sqrt{-1}.$$

This is a Fourier transform (with a sign change).

If $X \sim N(\mu, \sigma^2)$, then

$$\Phi_X(\omega) = \exp\left(i\omega\mu - \frac{\sigma^2\omega^2}{2}\right).$$

Since Fourier transform pairs are unique we get

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-i\omega x} d\omega.$$

**Definition:** The *moment generating function* of a random variable $X$ is given by

$$M_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx.$$

We may thus write

$$M_X(s) = \sum_{n=0}^{\infty} \frac{\mu_X^{(n)}}{n!} s^n, \quad \text{where } \mu_X^{(n)} = E\left[X^n\right]$$

provided all the moments are finite. We get

$$\mu_X^{(n)} = \left. \frac{d^n M_X(s)}{ds^n} \right|_{s=0}.$$

For $X \sim N(\mu, \sigma^2)$,

$$M_X(s) = \exp\left[\mu s + \frac{\sigma^2 s^2}{2}\right]$$

$$\left. \frac{dM_X(s)}{ds} \right|_{s=0} = \left. \left(\mu + \sigma^2 s\right) e^{\mu s + \sigma^2 s^2/2} \right|_{s=0} \Rightarrow \mu_X = \mu.$$

Also,

$$\left. \frac{d^2 M_X(s)}{ds^2} \right|_{s=0} = \left. \left(\mu + \sigma^2 s\right)^2 e^{\mu s + \sigma^2 s^2/2} + \sigma^2 e^{\mu s + \sigma^2 s^2/2} \right|_{s=0}$$

$$\Rightarrow \mu_X^{(2)} = \mu^2 + \sigma^2.$$

Thus,

$$Var(X) = \sigma^2.$$

## 3.2 Gaussian Random Vector

**Definition:** A random vector

$$\mathbf{W} = \left[W_1(u) \cdots W_n(u)\right]^t$$

is said to be a *white Gaussian random vector* if the random variables $W_1(u), \ldots, W_n(u)$ are independent and identically distributed (i.i.d.) Gaussian random variables with distribution $W_i \sim N(\mu, \sigma^2)$.

We will often make $\mu = 0$ and $\sigma^2 = 1$ since the data can always be translated and scaled so that this is so.

Thus,

$$\mu_W = (0, \ldots, 0)^t.$$

$$\mathbf{R_W} = \mathbf{K_W} = \sigma^2 \mathbf{I}_n.$$

$$f_{\mathbf{W}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{W_i}(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} \prod_{i=1}^{n} \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{||\mathbf{x}||^2}{2\sigma^2}\right)$$

where,

$$\mathbf{x}^t = (x_1, \ldots, x_n), \quad ||\mathbf{x}||^2 = \mathbf{x}^t \mathbf{x}.$$

**Definition:** A *Gaussian random vector* $\mathbf{X}(u)$ is defined as an affine transformation of the white Gaussian random vector.

We have

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{a}.$$

$$\mu_{\mathbf{X}} = E\left[\mathbf{H}\mathbf{W}(u) + \mathbf{a}\right] = \mathbf{a}.$$

$$\mathbf{K_X} = \mathbf{H}\mathbf{K_W}\mathbf{H}^\dagger = \mathbf{H}\sigma^2 \mathbf{I}_n \mathbf{H}^\dagger = \sigma^2 \mathbf{H}\mathbf{H}^\dagger.$$

If $\sigma^2 = 1$ then

$$\mathbf{K_X} = \mathbf{H}\mathbf{H}^\dagger.$$

## 3.3 Characteristic Function of a Gaussian Random Vector

Let

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{a}, \quad \mathbf{K_W} = \mathbf{I}_n.$$

Then

$$\Phi_X(\omega) = E\left[\exp\left\{i\omega^t \mathbf{X}\right\}\right], \quad \omega = (\omega_1, \ldots, \omega_n)^t$$

$$= E\left[\exp\left\{i\omega^t \left(\mathbf{H}\mathbf{W}(u) + \mathbf{a}\right)\right\}\right] = \exp\left\{i\omega^t \mathbf{a}\right\} \Phi_W(\mathbf{H}^t \omega).$$

Now

$$\Phi_W(\omega) = E\left[\exp\left\{i\omega^t \mathbf{W}(u)\right\}\right] = E\left[\exp\left\{i\sum_{k=1}^{n}\omega_k W_k(u)\right\}\right]$$

$$= \prod_{k=1}^{n}\Phi_{W_k}(\omega) = \exp\left\{-\frac{1}{2}\sigma^2||\omega||^2\right\} = \exp\left\{-\frac{1}{2}||\omega||^2\right\}.$$

Thus,

$$\Phi_X(\omega) = \exp\left\{i\omega^t \mu_{\mathbf{X}}\right\}\exp\left\{-\frac{1}{2}||\mathbf{H}^t\omega||^2\right\}$$

or

$$\Phi_X(\omega) = \exp\left\{i\omega^t\mu_{\mathbf{X}} - \frac{1}{2}\omega^t\mathbf{K}_{\mathbf{X}}\omega\right\}.$$

## 3.4 Density Function of a Gaussian Random Vector

$$\mathbf{X}(u) = \mathbf{H}\mathbf{W}(u) + \mathbf{a}.$$

Assume $\mathbf{H}$ is invertible. If

$$\mathbf{X} = g(\mathbf{W}) = (g_1(\mathbf{W}), \ldots, g_n(\mathbf{W}))^t$$

then

$$f_{\mathbf{X}}(\mathbf{x}) = \sum_k \frac{f_{\mathbf{W}}(\mathbf{y}_k)}{|J(\mathbf{y}_k)|}$$

where

$$g_i(\mathbf{y}_k) = x_i, \quad i = 1, \ldots, n, \quad \mathbf{y}_k = (y_{k,1}, \ldots, y_{k,n})^t, \quad \mathbf{x} = (x_1, \ldots, x_n)^t$$

and $J$ denotes the Jacobian of the transformation.

For us

$$\mathbf{X} = \mathbf{H}\mathbf{W} + \mathbf{a} = g(\mathbf{W}).$$

So we solve

$$\mathbf{x} = \mathbf{H}\mathbf{w} + \mathbf{a} \Rightarrow \mathbf{w} = \mathbf{H}^{-1}(\mathbf{x} - \mathbf{a}).$$

$$J(\mathbf{w}) = \mathbf{H}$$

so

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{det(\mathbf{H})} f_{\mathbf{W}}\left(\mathbf{H}^{-1}(\mathbf{x} - \mathbf{a})\right)$$

$$= \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}||\mathbf{H}^{-1}(\mathbf{x}-\mathbf{a})||^2\right\} \cdot \frac{1}{det(\mathbf{H})}.$$

Now

$$\mathbf{K_X} = \mathbf{H}\mathbf{H}^t \Rightarrow det(\mathbf{K_X}) = (det(\mathbf{H}))^2 .$$

So

$$det(\mathbf{H}) = \sqrt{det(\mathbf{K_X})}.$$

Thus,

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{det(\mathbf{K_X})}} \exp\left(-\frac{1}{2}\left(\mathbf{H}^{-1}(\mathbf{x}-\mathbf{a})\right)^\dagger \left(\mathbf{H}^{-1}(\mathbf{x}-\mathbf{a})\right)\right)$$

$$= \frac{1}{(2\pi)^{n/2}\sqrt{det(\mathbf{K_X})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{a})^\dagger \left(\mathbf{H}^{-1}\right)^\dagger \mathbf{H}^{-1}(\mathbf{x}-\mathbf{a})\right)$$

which becomes

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{det(\mathbf{K_X})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu_\mathbf{X})^\dagger \mathbf{K}_\mathbf{X}^{-1}(\mathbf{x}-\mu_\mathbf{X})\right).$$