EE 562a

Homework 9

(not to be handed in)

Work the following 5 problems.

Problem 1. Consider the mean square differential equation

$$\frac{dY(t)}{dt} + 2Y(t) = X(t)$$

for $t > 0$ subject to the initial condition $Y(0) = 0$. The input is

$$X(t) = e^{-t} + W(t)$$

where $W(t)$ is a white Gaussian noise process with mean zero and covariance function $K_W(\tau) = \sigma^2\delta(\tau)$.

a. Find $\mu_Y(t)$ for $t > 0$.

b. Find the covariance $K_Y(t_1, t_2)$ for $t_1, t_2 > 0$.

Solution.

a. The mean of $X(t)$ is e^{-t} so we need to solve the differential equation:

$$\frac{d\mu_Y(t)}{dt} + 2\mu_Y(t) = e^{-t}$$

with initial condition $\mu_Y(0) = 0$. The solution is:

$$\mu_Y(t) = e^{-t} - e^{-2t}, \quad t \geq 0.$$

b. We first solve $K_{YX}(t_1, t_2)$. The differential equation to be solved for $K_{YX}(t_1, t_2)$ is:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + K_{YX}(t_1, t_2) = K_X(t_1, t_2) = \sigma^2\delta(t_1 - t_2)$$

with initial condition $K_{YX}(0, t_2) = 0$. For $t_1 < t_2$ this becomes:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(0, t_2) = 0$ which has solution:

$$K_{YX}(t_1, t_2) = 0, \quad t_1 < t_2.$$

Next consider $t_1 > t_2$, with initial condition at $t_1 = t_2$. We need to solve:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_{YX}(t_1, t_2) = 0$$

with initial condition $K_{YX}(t_2, t_2) = \sigma^2$. This has solution:

$$K_{YX}(t_1, t_2) = \sigma^2 \exp[-2(t_1 - t_2)], \quad t_1 > t_2.$$

Putting all of these together:

$$K_{YX}(t_1, t_2) = \sigma^2 \exp[-2(t_1 - t_2)]u(t_1 - t_2).$$

Next, we solve for $K_Y(t_1, t_2)$, which has a differential equation:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) = K_{XY}(t_1, t_2) = K_{YX}(t_2, t_1).$$

For $0 < t_1 \leq t_2$, the equation is:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) = \sigma^2 \exp[-2(t_1 - t_2)]$$

with initial condition $K_Y(0, t_2) = 0$, which has solution:

$$K_Y(t_1, t_2) = \frac{\sigma^2}{4} e^{-2t_2}(e^{2t_1} - e^{-2t_1}).$$

For $t_1 > t_2$, we need to solve:

$$\frac{\partial K_{YX}(t_1, t_2)}{\partial t_1} + 2K_Y(t_1, t_2) = 0$$

with initial condition $K_Y(t_2, t_2) = \frac{\sigma^2}{4}(1 - e^{-4t_2})$, which has solution:

$$K_Y(t_1, t_2) = \frac{\sigma^2}{4}(1 - e^{-4t_2})e^{-2(t_1 - t_2)}.$$

Problem 2. Let $N(t)$ be a Poisson process with parameter λ. Find

$$E\left[(N(t) - N(s))^2\right]$$

for $t > s$.

Solution. We know that for a Poisson process,

$$\begin{aligned} E[N(t)] &= \lambda t \\ E[N(t)^2] &= \lambda t + \lambda^2 t^2. \end{aligned}$$

Moreover, for $t > s$,

$$\begin{aligned} E[N(t)N(s)] &= E[(N(s) + (N(t) - N(s)))(N(s))] \\ &= E[N(s)^2] + E[N(t) - N(s)]E[N(s)] \\ &= \lambda s + \lambda^2 s^2 + \lambda(t - s)\lambda s \\ &= \lambda s + \lambda^2 st. \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[(N(t) - N(s))^2\right] &= E[N(t)^2] + E[N(s)^2] - 2E[N(t)N(s)] \\ &= \lambda t + \lambda^2 t^2 + \lambda s + \lambda^2 s^2 - 2[\lambda s + \lambda^2 st] \\ &= \lambda t + \lambda s + \lambda^2 t^2 + \lambda^2 s^2 - 2\lambda^2 st. \end{aligned}$$

Problem 3. Let U be a random variable uniformly distributed on $(0, 1)$. Show that $1 - U$ is also uniformly distributed on $(0, 1)$.

Solution. For $x \in (0, 1)$, we have $P(U \leq x) = x$. Now

$$P(1 - U \leq x) = P(U \geq 1 - x) = 1 - P(U < 1 - x) = 1 - (1 - x) = x.$$

So, $1 - U$ is uniformly distributed on $(0, 1)$.

Problem 4. Show how to use the inverse transform method to generate a random variable X having density function

$$f(x) = \frac{e^x}{e - 1},\ 0 \leq x \leq 1.$$

Solution. For $0 \le x \le 1$

$$F(x) = \int_0^x \frac{e^x}{e-1}dy = \frac{e^x - 1}{e-1}.$$

Set $u = F(x)$ to get

$$u = \frac{e^x - 1}{e-1} \Rightarrow x = \ln(u(e-1)+1) \Rightarrow X = \ln(U(e-1)+1).$$

Problem 5. Let

$$\theta = \int_0^1 e^{x^2} dx.$$

a. Show how you can use two independent uniform random variables to estimte θ.

b. Show how you can use antithetic variables to estimte θ.

c. Show that the variance of the estimator in part (b) is less than the variance of the estimator in part (a).

Solution.
a. Let U be uniform in $(0,1)$. Then $\theta = E\left[e^{U^2}\right]$. So pick U_1 and U_2 independent and each uniform in $(0,1)$. Then

$$\hat{\theta} = \frac{e^{U_1^2} + e^{U_2^2}}{2}.$$

b. Let U be uniform in $(0,1)$. Thus $1-U$ is also uniform in $(0,1)$. Then

$$\hat{\theta} = \frac{e^{U^2} + e^{(1-U)^2}}{2}.$$

c. We find

$$\begin{aligned} Var(\hat{\theta}_2) &= \frac{1}{4} Var\left(e^{U^2} + e^{(1-U)^2}\right) \\ &= \frac{1}{4}\left(Var\left(e^{U^2}\right) + Var\left(e^{(1-U)^2}\right) + 2Cov\left(e^{U^2}, e^{(1-U)^2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
Var(\hat{\theta}_1) &= \frac{1}{4} Var\left(e^{U_1^2} + e^{U_2^2}\right) \\
&= \frac{1}{4}\left(Var\left(e^{U_1^2}\right) + Var\left(e^{U_2^2}\right)\right) \\
&= \frac{1}{4}\left(Var\left(e^{U^2}\right) + Var\left(e^{(1-U)^2}\right)\right)
\end{aligned}
$$

So, we jusy need to show $Cov\left(e^{U^2}, e^{(1-U)^2}\right) < 0$. Now,

$$
\begin{aligned}
Cov\left(e^{U^2}, e^{(1-U)^2}\right) &= E\left[e^{U^2} e^{(1-U)^2}\right] - E\left[e^{U^2}\right] E\left[e^{(1-U)^2}\right] < 0 \\
&\Leftrightarrow E\left[e^{U^2} e^{(1-U)^2}\right] < E\left[e^{U^2}\right] E\left[e^{(1-U)^2}\right]
\end{aligned}
$$

At this point, you can expand the exponential terms in a Taylor series, take expected values as indicated and the results will follow.