EE 562a

Homework 8

Due Wednesday July 29, 2009

Work the following 7 problems.

Problem 1. A zero mean sequence of i.i.d. random variables $x(n)$ form the input to a causal linear system defined by the linear difference equation

$$y(n) = \alpha y(n-1) + x(n), \quad |\alpha| < 1.$$

Find the impulse response $h(n)$ of a second system such that the sequence

$$z(n) = \sum_{i=0}^{k} h(i) y(n-i)$$

has a constant power spectral density, i.e.,

$$S_Z(f) = \sigma_z^2, \quad f \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Solution.

We are given a recursive definition of the sequence $y(n)$:

$$y(n) = \alpha y(n-1) + x(n),$$

where $x(n)$ is a sequence of zero mean i.i.d. random variables. Assume that $x(n)$ has unit variance so that $S_X(f) = 1$. Start by noting that:

$$y(n) = \sum_{i=0}^{\infty} \alpha^i x(n-i),$$

which means that $Y(z) = G(z)X(z)$ where:

$$G(z) = \frac{1}{1 - \alpha z^{-1}}.$$

We want to find $H(z)$ such that $H(z)G(z) = \sigma_Z$ so that $S_Z(f) = \sigma_Z^2$. We thus choose:

$$H(z) = \frac{\sigma_Z}{G(z)} = \sigma_Z(1 - \alpha^{-1}),$$

which gives the FIR filter:

$$h(n) = \sigma_Z(\delta_K(n) - \delta_K(n-1)).$$

Problem 2. A random process $X(t)$ is given by

$$X(t) = e^{-Yt}u(t)$$

where Y is a uniform random variable in the interval $[0,\ 1]$ and $u(t)$ is the unit step function. Find $P(X(t) \leq 0.25)$.

Solution.

If $t < 0$, then $X(t) = 0$ with probability 1. So, $Pr(X(t) \leq 0.25) = 1$.
If $t \geq 0$, then $u(t) = 1$. So:

$$\begin{aligned} Pr(X(t) \leq 0.25) &= Pr(e^{-Yt} \leq 0.25) \\ &= Pr(Y \geq \frac{\ln 4}{t}) \\ &= 1 - \frac{\ln 4}{t}. \end{aligned}$$

Problem 3. Let $X(t)$ be a stationary random process with mean μ_X and covariance function

$$K_X(\tau) = \frac{\sigma_X^2}{1+\tau^4}, \quad \infty < \tau < \infty.$$

a. Show that the mean square derivative exists for all t.

b. Find $\mu_{X'}(t)$ and $K_{X'}(\tau)$.

Solution.

a. Since μ_X is constant,

$$\begin{aligned} \frac{d^2 R_X(\tau)}{d\tau^2} &= \frac{d^2 K_X(\tau)}{d\tau^2} \\ &= \sigma_X^2 \left(\frac{36\tau^6}{(1+\tau^4)^3} - \frac{12\tau^2}{(1+\tau^4)^2} \right), \end{aligned}$$

which exists for all τ. So the mean-squared derivative of $X(t)$ exists for all τ.

b.

$$\mu_{X'}(t) = \frac{d}{dt}\mu_X(t) = 0.$$

$$\begin{aligned} K_{X'}(\tau) = R_{X'}(\tau) &= -\frac{d^2 R_X(\tau)}{d\tau^2} \\ &= \sigma_X^2 \left(-\frac{36\tau^6}{(1+\tau^4)^3} + \frac{12\tau^2}{(1+\tau^4)^2} \right). \end{aligned}$$

Problem 4. A stationary random process $X(t)$ has an autocorrelation function

$$R_X(\tau) = 10\exp(-|\tau|).$$

Show that $X(t)$ is ergodic in mean and in correlation.

Solution.

We know a WSS random process $X(t)$ is ergodic in mean if:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} K_X(\tau)(1 - \frac{|\tau|}{2T}) = 0.$$

Sufficient condition for this to occur is

$$\int_{-\infty}^{\infty} |R_X(\tau)| d\tau < \infty.$$

Hence, $X(t)$ is ergodic in mean since

$$\int_{-\infty}^{\infty} |10exp(-|\tau|)| d\tau = \int_{0}^{\infty} 10exp(-|\tau|) d\tau = 20 < \infty.$$

Problem 5. Give an example of a random process that is WSS but not ergodic in mean.

Solution.

A WSS process $X(t)$ has a constant mean and an correlation function $R_X(t_1, t_2)$ that depends only on $\tau = t_1 - t_2$. A process that is ergodic in mean has an absolutely integrable covariance function. Consider the process:

$$X(u,t) = \cos(\theta(u)),$$

where $\theta(u)$ is uniformly distributed on $[-\pi, \pi]$. In this case, $m_X(t) = 0$ and $R_x(t_1, t_2) = 1/2$. The process is WSS but not ergodic in mean since:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} K_X(\tau)(1 - \frac{|\tau|}{2T}) = 1,$$

which doesn't approach 0 when $T \to \infty$.

Problem 6. Let $X(t)$ be a WSS random process. Show that

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_x(t_1, t_2) = -\frac{d^2}{d\tau^2} R_x(\tau).$$

Solution.

$X(t)$ is WSS, so $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$. Thus:

$$\frac{\partial R_X(\tau)}{\partial \tau} = \frac{\partial R_X(t_1, t_2)}{\partial t_2} = -\frac{\partial R_X(t_1, t_2)}{\partial t_1}.$$

So,

$$\frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2} = -\frac{\partial^2 R_X(\tau)}{\partial \tau^2}.$$

Problem 7. Using the notation and definitions in the K-L theorem, let $X(t)$ be a random process and

$$\hat{X}(t) = \sum_{n=1}^{\infty} X_n \phi_n(t)$$

where

$$X_n = \int_{-T/2}^{T/2} X(t)\phi_n^*(t)dt.$$

Show (as stated in class) that

$$E\left[|\hat{X}(t) - X(t)|^2\right] = 0.$$

Solution.

We first determine $E[X_n(u)X_m^*(u)]$ and $E[X(u,t)X_n^*(u)]$ since they will be used in our proof:

$$\begin{aligned}
E[X_n(u)X_m^*(u)] &= E[\int_{-\frac{T}{2}}^{\frac{T}{2}} X(u,t)\phi_n^*(t)dt \int_{-\frac{T}{2}}^{\frac{T}{2}} X^*(u,s)\phi_m(s)ds] \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[X(u,t)\phi_n^*(t)X^*(u,s)\phi_m(s)]dtds \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n^*(t)K_X(t,s)\phi_m(s)dtds \\
&= \lambda_m \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n^*(t)\phi_m(s)dt \\
&= \lambda_m \delta_K[n,m].
\end{aligned}$$

$$\begin{aligned}
E[X(u,t)X_n^*(u)] &= E[X(u,t)\int_{-\frac{T}{2}}^{\frac{T}{2}} X^*(u,s)\phi_n(s)ds] \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} E[X(u,t)X^*(u,s)]\phi_n(s)ds \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} K_X(t,s)\phi_n(s)ds \\
&= \lambda_n\phi_n(t).
\end{aligned}$$

Now let:

$$\hat{X}_N(u,t) \triangleq \sum_{n=1}^{N} X_n(u)\phi_n(t).$$

We need to prove that:

$$\lim_{N\to\infty} E[|X(u,t)-\hat{X}_N(u,t)|^2]=0,$$

so let's look at the expression:

$$\begin{aligned}&E[|X(u,t)-\hat{X}_N(u,t)|^2]\\=&E[|X(u,t)|^2]-E[X(u,t)\hat{X}_N^*(u,t)]-E[X^*(u,t)\hat{X}_N(u,t)]+E[|\hat{X}_N(u,t)|^2].\end{aligned}$$

The first term:

$$E[|X(u,t)|^2]=K_X(t,t).$$

The second term:

$$\begin{aligned}E[X(u,t)\hat{X}_N^*(u,t)] &= E[X(u,t)\sum_{n=1}^{N}X_n^*(u)\phi_n^*(t)]\\ &=\sum_{n=1}^{N}E[X(u,t)\hat{X}_N^*(u,t)]\phi_n^*(t)\\ &=\sum_{n=1}^{N}\lambda_n\phi_n(t)\phi_n^*(t).\end{aligned}$$

The third term is the conjugate of the second:

$$E[X^*(u,t)\hat{X}_N(u,t)]=\sum_{n=1}^{N}\lambda_n\phi_n(t)\phi_n^*(t),$$

since $\{\lambda_n\}$ are real.
The fourth term:

$$\begin{aligned}E[|\hat{X}_N(u,t)|^2] &= E[\sum_{m=1}^{N}X_m(u)\phi_m(t)\sum_{n=1}^{N}X_n^*(u)\phi_n^*(t)]\\ &=\sum_{m=1}^{N}\sum_{n=1}^{N}E[X_m(u)X_n^*(u)]\phi_m(t)\phi_n^*(t)\\ &=\sum_{m=1}^{N}\sum_{n=1}^{N}\lambda_n\delta_K[m,n]\phi_m(t)\phi_n^*(t)\\ &=\sum_{n=1}^{N}\lambda_n\phi_n(t)\phi_n^*(t).\end{aligned}$$

The mean-squared distance between $X(u,t)$ and $\hat{X}_N(u,t)$ is thus:

$$E[|X(u,t) - \hat{X}_N(u,t)|^2] = K_X(t,t) - \sum_{n=1}^{N} \lambda_n \phi_n(t) \phi_n^*(t).$$

From Mercer's theorem, as $N \to \infty$, this mean-square distance tends to 0.