EE 562a

Homework 7

Due Friday July 24, 2009

Work the following 10 problems.

Problem 1. Which of the following functions of real variables t_1 and t_2 are non-negative Hermitian symmetric (and hence could be covariance functions of a random process whose index set T is the real line). Give proofs in each case.

a. $\sin(t_1 - t_2)$.
Since

$$\sin^*(t_1 - t_2) = \sin(t_2 - t_1) \neq \sin(t_1 - t_2)$$

Function is not Hermitian symmetric and so is not a valid covariance function.

b. $2 + \cos(t_1 - t_2)$.
Function is Hermitian symmetric. Check for non-negative definiteness.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^*(2+\cos(t_i - t_j)) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^*(2+\cos t_i \cos t_j + \sin t_i \sin t_j) \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* + \sum_{i=1}^{n} a_i \cos t_i \sum_{j=1}^{n} a_j^* \cos t_j \\
&\quad + \sum_{i=1}^{n} a_i \sin t_i \sum_{j=1}^{n} a_j^* \sin t_j \\
&= 2\left|\sum_{i=1}^{n} a_i\right|^2 + \left|\sum_{i=1}^{n} a_i \cos t_i\right|^2 + \left|\sum_{i=1}^{n} a_i \sin t_i\right|^2 \\
&\geq 0.
\end{aligned}$$

Function is non-negative definite and Hermitian symmetric and so is a valid covariance function.

c. $e^{t_1 - t_2}$.
Function is not Hermitian symmetric and so is not a valid covariance function.

d. $e^{-t_1-t_2}$.

Function is Hermitian symmetric. Check for non-negative definiteness.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* e^{-t_i-t_j} &= \sum_{i=1}^{n} a_i e^{-t_i} \sum_{j=1}^{n} a_j^* e^{-t_j} \\ &= \left[\sum_{i=1}^{n} a_i e^{-t_i}\right]\left[\sum_{j=1}^{n} a_j e^{-t_j}\right]^* \\ &= \left|\sum_{i=1}^{n} a_i e^{-t_i}\right|^2 \\ &\geq 0.\end{aligned}$$

e. $e^{i\omega(-t_1-t_2)}$.

Function is not Hermitian symmetric and so is not a valid covariance function.

Problem 2. Consider a random sequence $x(n)$ with zero mean and covariance

$$K_X(m,n) = a^{|m-n|}.$$

A second sequence $y(n)$ is generated as

$$y(n) = x(n) - x(n-1) - x(n-2).$$

a. Compute the cross covariance of x and y and the corresponding cross spectral density.

b. Compute the covariance of y and its power spectral density.

Solution.

To answer this question, we can make use of the DTFT pair:

$$\rho^{-|\tau|} \Longleftrightarrow \frac{1-\rho^2}{\left(1-\rho e^{-j2\pi f}\right)\left(1-\rho e^{j2\pi f}\right)}.$$

Using this pair, we find the power spectral density of $x(n)$:

$$S_x(f) = \frac{a^2-1}{\left(a-e^{-j2\pi f}\right)\left(a-e^{j2\pi f}\right)}.$$

a.

$$S_{XY^*}(f) = H(f)^* S_X(f) = (1 - e^{j2\pi f} - e^{j4\pi f}) S_X(f).$$

$$\begin{aligned} K_{XY^*}(\tau) = \mathcal{F}^{-1}\left(S_{XY^*}(f)\right) &= K_X(\tau) - K_X(\tau+1) - K_X(\tau+2) \\ &= a^{|\tau|} - a^{|\tau+1|} - a^{|\tau+2|}, \end{aligned}$$

where $\tau = m - n$.

b.

$$\begin{aligned} S_Y(f) = |H(f)|^2 S_X(f) &= \left(3 - e^{j4\pi f} - e^{-j4\pi f}\right) S_X(f) \\ &= \left(3 - 2\cos(4\pi f)\right) S_X(f). \end{aligned}$$

$$\begin{aligned} K_Y(\tau) = \mathcal{F}^{-1}\left(S_Y(f)\right) &= 3K_X(\tau) - K_X(\tau+2) - K_X(\tau-2) \\ &= 3a^{|\tau|} - a^{|\tau+2|} - a^{|\tau-2|}. \end{aligned}$$

Problem 3. Use the orthogonality principle to derive the optimal linear estimator (Wiener filter) of $x(n)$ given observations of the random sequence $y(n)$ formed according to the equation

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) + w(n)$$

where $w(n)$ and $x(n)$ are mutually independent, zero mean, wide sense stationary random sequences with known covariances and $h(n)$ is the known impulse response of a shift invariant, causal, stable system.

Solution.

The Wiener filter estimate of $x(n)$ from $y(n)$ has the form:

$$\hat{x}(n) = \sum_{k=-\infty}^{\infty} g(k)y(n-k).$$

By orthogonality principle, the error signal $e(n) = x(n) - \hat{x}(n)$ is orthogonal to $\hat{x}(n)$, thus:

$$
\begin{aligned}
E\left[(x(n) - \hat{x}(n))\hat{x}(n)\right] = \sum_{k=-\infty}^{\infty} g(k) E\left[y(n-k)x(n)\right] - \\
\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g(k)g(l) E\left[y(n-k)y(n-l)\right] \\
= \sum_{k=-\infty}^{\infty} g(k) \left[K_{XY}(k) - \sum_{l=-\infty}^{\infty} g(l) K_Y(k-l) \right] = 0.
\end{aligned}
$$

So, for all k, we can simply let:

$$
K_{XY} = \sum_{l=-\infty}^{\infty} g(l) K_Y(k-l).
$$

Taking Fourier Transform:

$$
\begin{aligned}
S_{XY}(f) &= G(f) S_Y(f) \\
G(f) &= \frac{S_{XY}(f)}{S_Y(f)}.
\end{aligned}
$$

Problem 4. Suppose that you can observe a signal which is the sum of three mutually orthogonal, WSS, zero-mean random sequences which is passed through a known linear system, i.e.,

$$
y(n) = \sum_{k=0}^{N} h(k)[x_1(n-k) + x_2(n-k) + x_3(n-k)].
$$

Assume their covariances are known.

a. Construct a Wiener filter to optimally recover the sequence $x_1(n)$ from $y(n)$.

b. Suppose that the signal is corrupted with additive noise, i.e.,

$$
z(n) = y(n) + w(n)
$$

where $w(n)$ is WSS and uncorrelated with the sequences $x_1(n)$,$x_2(n)$ and $x_3(n)$. Find an optimal Wiener filter to recover $x_1(n)$ from $z(n)$.

Solution.

a. First note that because $x_1(n)$, $x_2(n)$ and $x_3(n)$ are mutually orthogonal, for any n and m,

$$E[x_1(n)x_2(m)] = 0$$
$$E[x_1(n)x_3(m)] = 0.$$

From Q3, we know the form of the Wiener filter is

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)}.$$

So, we need to calculate $S_{X_1Y}(f)$ and $S_Y(f)$:

$$\begin{aligned}
K_{X_1Y}(m) &= E[x_1(n)y(n-m)] \\
&= E\left[x_1(n)\sum_{k=0}^{N} h(k)[x_1(n-m-k) + x_2(n-m-k) + x_3(n-m-k)]\right] \\
&= \sum_{k=0}^{N} h(k)K_{X_1}(m+k).
\end{aligned}$$

Hence,

$$S_{X_1Y}(f) = S_{X_1}(f)H^*(f).$$

$$\begin{aligned}
K_Y(m) &= E[y(n)y(n-m)] \\
&= E[\sum_{k=0}^{N} h(k)[x_1(n-m-k) + x_2(n-m-k) + x_3(n-m-k)] \\
&\quad \sum_{l=0}^{N} h(l)[x_1(n-m-l) + x_2(n-m-l) + x_3(n-m-l)]] \\
&= \sum_{k=0}^{N}\sum_{l=0}^{N} h(k)h(l)K_{X_1}(l+m-k) + \\
&\quad \sum_{k=0}^{N}\sum_{l=0}^{N} h(k)h(l)K_{X_2}(l+m-k) + \\
&\quad \sum_{k=0}^{N}\sum_{l=0}^{N} h(k)h(l)K_{X_3}(l+m-k)
\end{aligned}$$

Hence,

$$S_Y(f) = S_{X_1}(f)|H(f)|^2 + S_{X_2}(f)|H(f)|^2 + S_{X_3}(f)|H(f)|^2.$$

Finally, the Wiener filter to recover $x_1(n)$ from $y(n)$ is

$$G(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{S_{X_1}(f)H^*(f)}{[S_{X_1}(f) + S_{X_2}(f) + S_{X_3}(f)]|H(f)|^2}.$$

b. Here we need $S_{X_1Z}(f)$ and $S_Z(f)$. Because $w(n)$ is uncorrelated with the zero-mean sequences $x_1(n)$,$x_2(n)$ and $x_3(n)$:

$$K_{X_1Z}(m) = K_{X_1Y}(m)$$
$$S_{X_1Z}(f) = S_{X_1Y}(f),$$

and

$$K_Z(m) = K_Y(m) + K_W(m)$$
$$S_Z(f) = S_Y(f) + K_W(f).$$

The Wiener filter to recover $x_1(n)$ from $z(n)$ is thus,

$$G(f) = \frac{S_{X_1}(f)H^*(f)}{[S_{X_1}(f) + S_{X_2}(f) + S_{X_3}(f)]|H(f)|^2 + S_W(f)}.$$

Problem 5. The power spectral density of a random sequence $x(n)$ is

$$S_x(f) = \frac{1}{(1 + a^2) - 2a\cos(2\pi f)}.$$

a. Find the covariance function of the sequence $x(n)$.

b. Find the frequency response of a filter which will produce a sequence with this PSD with an input which is an i.i.d. random sequence.

c. Find the linear difference equation for this system.

Solution.

Simplifying $S_x(f)$ and converting to Z-domain,

$$S_X(z) = \frac{1}{(1 - az)(1 - az^{-1})}.$$

a. Assume $|a| < 1$ and a is real. There is only one pole inside the unit circle at $z = a$. We find the inverse Z-transform using the Residue theorem. For $m \geq 0$,

$$\begin{aligned} K_X(m) &= \oint S_X(z) z^{m-1} dz \\ &= \text{Res} \left[S_X(z) z^{m-1} |_{z=a} \right] \\ &= \frac{a^m}{1-a^2}. \end{aligned}$$

For $m < 0$, we know that $K_X(m) = K_X(-m)$, so for all m, we have:

$$K_X(m) = \frac{a^{|m|}}{1-a^2}.$$

b. Assume i.i.d. input sequence $w[n]$ has unit variance. We need to solve:

$$S_X(z) = \frac{1}{(1-az)(1-az^{-1})} = H(z)H^*(z^{-1}).$$

Since we assume $|a| < 1$, the simulation filter is:

$$H(z) = \frac{1}{1-az^{-1}}.$$

c. $X(z) - az^{-1}X(z) = W(z)$. So the linear difference equation for this system is:

$$x[n] - ax[n-1] = w[n].$$

Problem 6. Find a causal method of constructing a wide-sense stationary random sequence $X(u, n)$ with spectral density

$$S_x(f) = \frac{\frac{5}{4} - \cos(2\pi f)}{\left(\frac{13}{4} - 3\cos(2\pi f + \pi/4)\right)\left(\frac{13}{4} - 3\cos(2\pi f - \pi/4)\right)}$$

and draw a simple block diagram indicating a mechanization of the generation process.

Solution.

Simplifying $S_x(f)$ and converting to Z-domain,

$$S_x(f) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}{\left((1-\frac{3}{2}e^{j\frac{\pi}{4}}z)(1-\frac{3}{2}e^{-j\frac{\pi}{4}}z^{-1})(1-\frac{3}{2}e^{-j\frac{\pi}{4}}z)(1-\frac{3}{2}e^{j\frac{\pi}{4}}z^{-1})\right)}.$$

We choose the poles and zeroes inside the unit circle to get:

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{3}{2}e^{j\frac{\pi}{4}}z)(1-\frac{3}{2}e^{-j\frac{\pi}{4}}z)}.$$

We can filter an unit variance i.i.d. sequence $w(n)$ with $H(z)$ to generate $x(n)$ with power spectral density $S_x(f)$.

Problem 7. Consider the following two random processes

$$X(u,t) = 1 \text{ for all } u \in U,\ t \in T$$

$$Y(u,t) = A(u) \text{ for all } u \in U,\ t \in T$$

where $A(u)$ is a zero-mean, unit variance random variable.

a. Show that $X(u,t)$ and $Y(u,t)$ are wide-sense stationary.

b. Find the power spectral densities $S_x(f)$ and $S_y(f)$.

Solution.

a. Since $X(u,t)$ is constant with probability 1, $E[X(u,t)] = 1$, which is a constant independent of t. And $R_X(t,s) = 1$, which is a constant function of $\tau = t - s$. So, $X(u,t)$ is w.s.s.. Similarly, $E[Y(u,t)] = E[A(u)] = 0$ and $R_Y(t,s) = E[Y(u,t)Y(u,s)^*] = E[|A(u)|^2] = 1$, so $Y(u,t)$ is w.s.s..

b. $R_X(\tau) = R_Y(\tau) = 1$, so $S_X(f) = S_Y(f) = \delta_D(f)$.

Problem 8. Consider a random process $X(t)$ defined by

$$X(t) = U\cos\omega t + V\sin\omega t$$

where ω is a constant and U and V are random variables.

a. Show the condition

$$E[U] = E[V] = 0$$

is necessary for the random process to be stationary.

b. Show that $X(t)$ is wide sense stationary if and only if U and V are uncorrelated with equal variance σ^2.

c. Now let $\omega = 1$ and assume U and V are independent random variables each of which assume the values -2 and 1 with probabilities 1/3 and 2/3, respectively. Show $X(t)$ is WSS but not strict-sense stationary.

Solution.

a. For $X(t)$ to be w.s.s., its mean function should be a constant. But,

$$E[X(t)] = E[U]\cos\omega t + E[V]\sin\omega t$$

can only be constant if $E[U] = E[V] = 0$.

b.

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[U^2]\cos(\omega t_1)\cos(\omega t_2) + E[V^2]\sin(\omega t_1)\sin(\omega t_2) \\
&\quad + E[UV]\left(\sin(\omega t_1)\cos(\omega t_2) + \cos(\omega t_1)\sin(\omega t_2)\right) \\
&= \frac{1}{2}\cos(\omega(t_1 + t_2))(E[U^2] - E[V^2]) + \frac{1}{2}\cos(\omega(t_1 - t_2))(E[U^2] + E[V^2]) \\
&\quad + E[UV]\sin(\omega(t_1 + t_2)).
\end{aligned}$$

If U and V are uncorrelated and $E[U] = E[V] = 0$ (part a), then $E[UV] = E[U]E[V] = 0$. In addition, if $E[U^2] = E[V^2] = \sigma^2$, then:

$$\begin{aligned}
E[X(t)] &= 0 \\
R_X(t_1, t_2) &= \cos(w(t_1 - t_2)),
\end{aligned}$$

i.e. $X(t)$ is wide sense stationary.

c. It is easy to check U and V satisfy the sufficient conditions for w.s.s.

in part b. To show $X(t)$ is not strictly stationary, we find a counter-example. Let $t_1 = 0$,$t_2 = \frac{\pi}{2\omega}$ and $\tau = \frac{\pi}{2\omega}$. Then,

$$F_{X(t_1),X(t_2)}(0,0) = Pr(U \leq 0)Pr(V \leq 0) = \frac{1}{9}$$

$$F_{X(t_1+\tau),X(t_2+\tau)}(0,0) = Pr(V \leq 0)Pr(-U \leq 0) = \frac{2}{9}.$$

Since, $F_{X(t_1),X(t_2)}(0,0) \neq F_{X(t_1+\tau),X(t_2+\tau)}(0,0)$, $X(t)$ is not strictly stationary.

Problem 9. Let $x(n)$ be an independent and identically distributed random sequence with each $x(n)$ having mean 0 and variance σ^2. Suppose we form

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

where,

$$h(n) = \begin{cases} (-\alpha)^{n/2}, & n \geq 0, \\ 0, & \text{elsewhere} \end{cases}$$

where $0 < \alpha < 1$. Find $S_y(f)$, the power spectral density of Y.

Solution.

$$S_Y(f) = |H(f)|^2 S_X(f)$$

where $S_X(f) = \sigma^2$ and

$$H(z) = \sum_{m=0}^{\infty} h(m)z^{-m} = \sum_{n=0}^{\infty} h(2n)z^{-2n}$$

$$= \sum_{n=0}^{\infty} (-\alpha z^{-2})^n = \frac{1}{1+\alpha z^{-2}}.$$

So,

$$H(f) = \frac{1}{1+\alpha e^{-4\pi f n}}$$

$$\Rightarrow |H(f)|^2 = \frac{1}{1+\alpha^2+2\alpha\cos(4\pi f n)}$$

$$\Rightarrow S_y(f) = \frac{\sigma^2}{1+\alpha^2+2\alpha\cos(4\pi f n)}.$$

Problem 10. Say we have a random process

$$Z(t) = XY\cos(2\pi t + \theta)$$

where X and Y are independent of θ with $\theta \sim U[-\pi, \pi]$ and X and Y are jointly distributed as

$$f_{XY}(x,y) = \begin{cases} 1/2\pi, & (x,y) \in D \\ 0, & \text{elsewhere} \end{cases}$$

where, D is the region in the plane bounded by the x-axis and the semi-circle described for positive y by $y = \sqrt{4-x^2}$, $-2 \le x \le 2$.

Suppose we learn that $X = x$, i.e., we have knowledge of the random variable X. Under these conditions find the mean function and the covariance function of $Z(t)$.

Solution.

$$E[Z(t)|X = x] = xE[Y|X = x]E[\cos(2\pi t + \theta)] = 0.$$

Similarly,

$$\begin{aligned} K_Z(t_1, t_2) &= E[Z(t_1)Z(t_2)|X = x] \\ &= \frac{x^2}{2}E[Y^2|X = x]\cos(2\pi\tau). \end{aligned}$$

where $\tau = t_1 - t_2$. Since,

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ f_X(x) &= \int_0^{\sqrt{4-x^2}} \frac{1}{2\pi}dy = \frac{1}{2\pi}\sqrt{4-x^2}, \quad -2 \le x \le 2 \\ \Rightarrow f_{Y|X}(y|x) &= \frac{1}{\sqrt{4-x^2}} \end{aligned}$$

So,

$$\begin{aligned} K_Z(\tau) &= \frac{x^2}{2}\cos(2\pi\tau)\int_0^{\sqrt{4-x^2}} \frac{y^2}{\sqrt{4-x^2}}dy \\ &= \frac{(4-x^2)x^2\cos(2\pi\tau)}{6}. \end{aligned}$$