EE 562a

Homework 7

Due Friday July 24, 2009

Work the following 10 problems.

Problem 1. Which of the following functions of real variables t_1 and t_2 are non-negative Hermitian symmetric (and hence could be covariance functions of a random process whose index set T is the real line). Give proofs in each case.

a. $\sin(t_1 - t_2)$.

b. $2 + \cos(t_1 - t_2)$.

c. $e^{t_1 - t_2}$.

c. $e^{-t_1 - t_2}$.

d. $e^{i\omega(-t_1 - t_2)}$.

Problem 2. Consider a random sequence $x(n)$ with zero mean and covariance

$$K_X(m, n) = a^{|m-n|}.$$

A second sequence $y(n)$ is generated as

$$y(n) = x(n) - x(n-1) - x(n-2).$$

a. Compute the cross covariance of x and y and the corresponding cross spectral density.

b. Compute the covariance of y and its power spectral density.

Problem 3. Use the orthogonality principle to derive the optimal linear estimator (Wiener filter) of $x(n)$ given observations of the random sequence $y(n)$ formed according to the equation

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) + w(n)$$

where $w(n)$ and $x(n)$ are mutually independent, zero mean, wide sense stationary random sequences with known covariances and $h(n)$ is the known impulse response of a shift invariant, causal, stable system.

Problem 4. Suppose that you can observe a signal which is the sum of three mutually orthogonal, WSS, zero-mean random sequences which is passed through a known linear system, i.e.,

$$y(n) = \sum_{k=0}^{N} h(k)[x_1(n-k) + x_2(n-k) + x_3(n-k)].$$

Assume their covariances are known.

a. Construct a Wiener filter to optimally recover the sequence $x_1(n)$ from $y(n)$.

b. Suppose that the signal is corrupted with additive noise, i.e.,

$$z(n) = y(n) + w(n)$$

where $w(n)$ is WSS and uncorrelated with the sequences $x_1(n)$, $x_2(n)$ and $x_3(n)$. Find an optimal Wiener filter to recover $x_1(n)$ from $z(n)$.

Problem 5. The power spectral density of a random sequence $x(n)$ is

$$S_x(f) = \frac{1}{(1+a^2) - 2a\cos(2\pi f)}.$$

a. Find the covariance function of the sequence $x(n)$.

b. Find the frequency response of a filter which will produce a sequence with this PSD with an input which is an i.i.d. random sequence.

c. Find the linear difference equation for this system.

Problem 6. Find a causal method of constructing a wide-sense stationary random sequence $X(u,n)$ with spectral density

$$S_x(f) = \frac{\frac{5}{4} - \cos(2\pi f)}{\left(\frac{13}{4} - 3\cos(2\pi f + \pi/4)\right)\left(\frac{13}{4} - 3\cos(2\pi f - \pi/4)\right)}$$

and draw a simple block diagram indicating a mechanization of the generation process.

Problem 7. Consider the following two random processes

$$X(u,t) = 1 \text{ for all } u \in U,\ t \in T$$

$$Y(u,t) = A(u) \text{ for all } u \in U,\ t \in T$$

where $A(u)$ is a zero-mean, unit variance random variable.

a. Show that $X(u,t)$ and $Y(u,t)$ are wide-sense stationary.

b. Find the power spectral densities $S_x(f)$ and $S_y(f)$.

Problem 8. Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t$$

where ω is a constant and U and V are random variables.

a. Show the condition

$$E[U] = E[V] = 0$$

is necessary for the random process to be stationary.

b. Show that $X(t)$ is wide sense stationary if and only if U and V are uncorrelated with equal variance σ^2.

c. Now let $\omega = 1$ and assume U and V are independent random variables each of which assume the values -2 and 1 with probabilities 1/3 and 2/3, respectively. Show $X(t)$ is WSS but not strict-sense stationary.

Problem 9. Let $x(n)$ be an independent and identically distributed random sequence with each $x(n)$ having mean 0 and variance σ^2. Suppose we form

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

where,

$$h(n) = \begin{cases} (-\alpha)^{n/2}, & n \geq 0, \\ 0, & \text{elsewhere} \end{cases}$$

where $0 < \alpha < 1$. Find $S_y(f)$, the power spectral density of Y.

Problem 10. Say we have a random process

$$Z(t) = XY\cos(2\pi t + \theta)$$

where X and Y are independent of θ with $\theta \sim U[-\pi, \pi]$ and X and Y are jointly distributed as

$$f_{XY}(x, y) = \begin{cases} 1/2\pi, & (x, y) \in D \\ 0, & \text{elsewhere} \end{cases}$$

where, D is the region in the plane bounded by the x-axis and the semi-circle described for positive y by $y = \sqrt{4 - x^2}$, $-2 \leq x \leq 2$.

Suppose we learn that $X = x$, i.e., we have knowledge of the random variable X. Under these conditions find the mean function and the covariance function of $Z(t)$.