EE 562a

Homework 6

Due Wednesday July 15, 2009

Work the following 7 problems.

Problem 1. Let $X_1, \ldots X_n$ be n random variables each with mean μ and variance $\sigma^2 < \infty$. Let us estimate the variance as

$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

a. Show that $\hat{\sigma^2}$ is an unbiased estimator for σ^2.

b. Show that $\hat{\sigma^2} \to \sigma^2$ in probability.

Solution.

a.

$$\begin{aligned}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 &= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2\right] \\ &= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]\end{aligned}$$

Hence,

$$\begin{aligned}E[\hat{\sigma}^2] &= \frac{1}{n-1}\left[nE[X_i^2] - nE[\bar{X}^2]\right] \\ &= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] \\ &= \sigma^2.\end{aligned}$$

b. First,

$$\hat{\sigma}^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

$$
\begin{aligned}
&= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - 2n\mu\bar{X} + n\mu^2 - (n\bar{X}^2 - 2n\mu\bar{X} + n\mu^2)\right] \\
&= \frac{1}{n-1}\left[\sum_{i=1}^{n} (X_i-\mu)^2 - n(\bar{X}-\mu)^2\right].
\end{aligned}
$$

By the strong law of large numbers,

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 &\to E[(X_i-\mu)^2] = Var(X) = \sigma^2 \ \text{ a.s.} \\
\text{and } \bar{X} &\to \mu \ \text{ a.s.}
\end{aligned}
$$

Hence,

$$
\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 - (\bar{X}-\mu)^2\right] \to \sigma^2 \ \text{ a.s.}
$$

Since $\hat{\sigma}^2 \to \sigma^2$ a.s., therefore $\hat{\sigma}^2 \to \sigma^2$ in Probability.

Problem 2. Give an example of two random variables X and Y in which the best linear predictor of Y given X is a constant (has no predictive value) whereas the best predictor of Y given X predicts Y perfectly (without error).

Solution.

An example is $Y = X^2$ where $X \sim Uniform(-1,1)$. The linear predictor is $\hat{Y} = aX + b$, with

$$
\begin{aligned}
a &= \frac{\sigma_{XY}}{\sigma_X^2} \\
b &= \mu_Y - a\mu_X
\end{aligned}
$$

But, $\mu_X = E[X] = 0$ and

$$
E[XY] = E[X^3] = \frac{1}{2}\int_{-1}^{1} x^3 dx = 0
$$

Hence, $\sigma_{XY} = cov(X,Y) = E[XY] - E[X]E[Y] = 0$. Therefore $a = 0$ and $b = E[Y]$ and the best linear predictor gives $\hat{Y} = b =$constant.

However, the best estimator

$$\hat{Y} = E[Y|X] = E[X^2|X] = X^2 = Y$$

That is, the best predictor of Y given X predicts Y perfectly.

Problem 3. For the following pairs of data find the best least squares fit for a model of the form

$$z_i = a + bx_i + \epsilon_i,$$

with $E[\epsilon_i = 0]$, $Var(\epsilon_i) = \sigma^2$, $E[\epsilon_i \epsilon_j] = 0$ for $i \neq j$, that is, find the regression line $a + bx$.

$$(x_i, z_i) =$$

$(1.0, 4.5), (1.1, 5.3), (1.5, 6.2), (2.0, 6.4), (3.2, 9.0), (4.0, 10.1), (4.5, 11.5), (5.0, 13.3).$

Solution.

The data can be arranged into matrix form $Ax = b$, where

$$A \triangleq \begin{bmatrix} 1 & 1.0 \\ 1 & 1.1 \\ 1 & 1.5 \\ 1 & 2.0 \\ 1 & 3.2 \\ 1 & 4.0 \\ 1 & 4.5 \\ 1 & 5.0 \end{bmatrix} \quad x \triangleq \begin{bmatrix} a \\ b \end{bmatrix} \quad b \triangleq \begin{bmatrix} 4.5 \\ 5.3 \\ 6.2 \\ 6.4 \\ 9.0 \\ 10.1 \\ 11.5 \\ 13.3 \end{bmatrix}$$

The best solution $\hat{x} \triangleq (\hat{a}, \hat{b})$ that minimizes the squared error can be found from

$$\begin{aligned} A^T A\hat{x} &= A^T b \\ \hat{x} &= (A^T A)^{-1} A^T b = [2.7922, 1.9714]^T. \end{aligned}$$

That is, $a = 2.7922$ and $b = 1.9714$. Plotting the results in a graph:

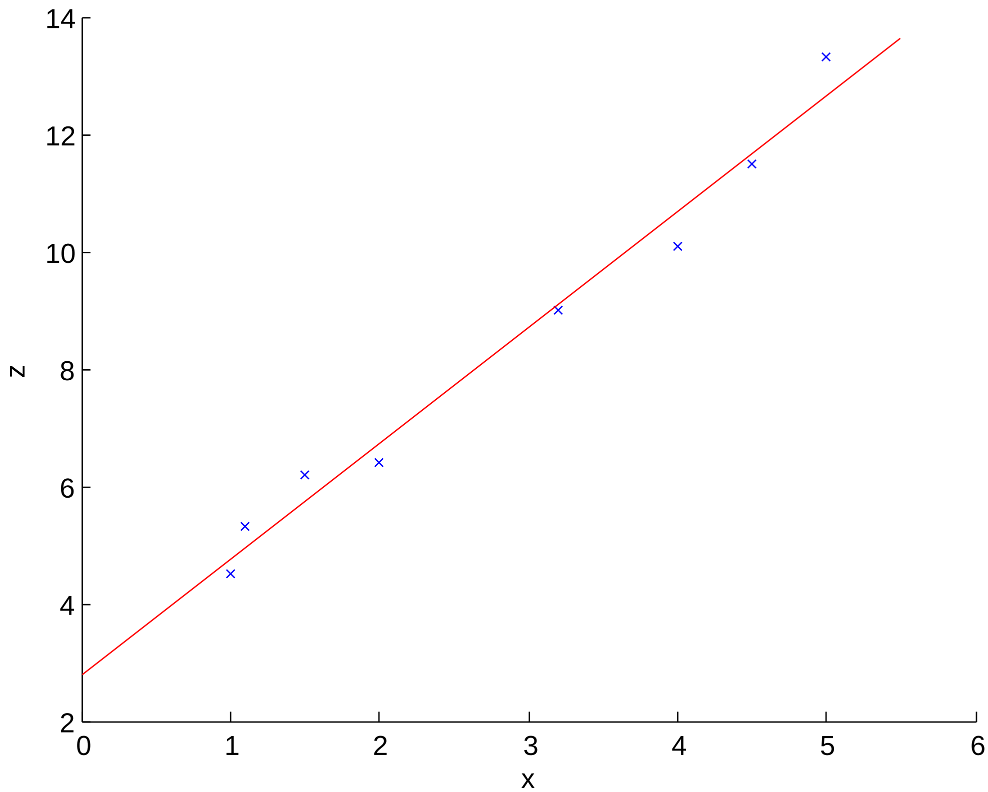

Figure 1: Best Linear Least Squares fit for given data.

Problem 4. Consider the random process

$$X(u,t) = Be^{-A(u)t}$$

where $A(u)$ is a random variable with probability density function

$$f_A(x) = \begin{cases} ce^{-cx}, & c > 0,\ x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean, correlation and covariance functions of $X(u,t)$.

Solution.

$$m_X(t) \;\; = \;\; E[X(u,t)]$$

$$
\begin{aligned}
&= Bc\int_0^\infty e^{-(c+t)x}dx \\
&= \frac{B}{1+\frac{t}{c}}.
\end{aligned}
$$

$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(u,t_1)X^*(u,t_2)] \\
&= B^2E\left[e^{-A(u)(t_1+t_2)}\right] \\
&= B^2c\int_0^\infty e^{-(t_1+t_2+c)x}dx \\
&= \frac{B^2}{1+\frac{t_1+t_2}{c}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
K_X(t_1,t_2) &= R_X(t_1,t_2) - m_X(t_1)m_X^*(t_2) \\
&= \frac{B^2}{1+\frac{t_1+t_2}{c}} - \frac{B}{1+\frac{t_1}{c}}\frac{B}{1+\frac{t_2}{c}} \\
&= B^2\left[\frac{1}{1+\frac{t_1+t_2}{c}} - \frac{1}{(1+\frac{t_1}{c})(1+\frac{t_2}{c})}\right].
\end{aligned}
$$

Problem 5. Find the mean and covariance functions for the random process

$$X(u,t) = A(u,t)\cos(2\pi f_0 t + \phi(u))$$

where $\phi(u)$ is a random variable uniformly distributed on the interval $(-\pi,\ \pi)$ and is independent of the amplitude modulation $A(u,t)$. Assume the correlation function, $R_A(t_1,t_2)$, of the modulation is known.

Solution.

$$
\begin{aligned}
m_X(t) &= E[A(u,t)]E[\cos(2\pi f_0 t + \phi(u))] \\
&= \frac{1}{2\pi}E[A(u,t)]\int_{-\pi}^{\pi}\cos(2\pi f_0 t + \phi)d\phi = 0.
\end{aligned}
$$

$$
\begin{aligned}
K_X(t_1,t_2) &= E[A(u,t_1)A^*(u,t_2)\cos(2\pi f_0 t_1 + \phi(u))\cos(2\pi f_0 t_2 + \phi(u))] \\
&= \frac{1}{2}E[A(u,t_1)A^*(u,t_2)]\left\{E[\cos(2\pi f_0(t_1-t_2)] + E[\cos(2\pi f_0(t_1+t_2) + 2\phi(u))]\right\} \\
&= \frac{1}{2}R_A(t_1,t_2)\cos(2\pi f_0(t_1-t_2).
\end{aligned}
$$

Problem 6. Consider the complex random process

$$X(u,t) = e^{i2\pi f(u)t}$$

where $f(u)$ is a uniform random variable on $\Omega = [f_0 - \Delta f, f_0 + \Delta f]$, i.e., the sample paths of the random process are complex sinusoids (or exponentials) with frequencies in the range $f_0 \pm \Delta f$. The index set T is the real line. Compute the mean and covariance functions of the random process.

Solution.

$$\begin{aligned} m_X(t) &= E[e^{i2\pi f(u)t}] \\ &= \frac{1}{2\Delta f}\int_{f_0-\Delta f}^{f_0+\Delta f} e^{i2\pi f t} df \\ &= \frac{1}{2\pi\Delta f t} e^{i2\pi f_0 t}\sin(2\pi\Delta f t). \end{aligned}$$

$$\begin{aligned} R_X(t_1,t_2) &= E[e^{i2\pi f(u)(t_1-t_2)}] \\ &= \frac{1}{2\pi\Delta f(t_1-t_2)} e^{i2\pi f_0(t_1-t_2)}\sin(2\pi\Delta f(t_1-t_2)). \end{aligned}$$

$$K_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X^*(t_2).$$

Problem 7. Let $w(n)$ be an independent random sequence with mean zero and variance σ^2. Define the new random sequence $x(n)$ as

$$x(0) = 0,$$

$$x(n) = \rho x(n-1) + w(n), \quad n \geq 1.$$

a. Find the mean of $x(n)$ for $n \geq 0$.

b. Find the covariance of $x(n)$, denoted $K_X(m,n)$.

c. For what values of ρ does $K_X(m,n)$ converge to some finite valued function $g(m-n)$ as m and n become large (this is called *asymptotic stationarity*).

Solution.

First note that $x(n) = \rho x(n-1) + w(n)$ with $x(0) = 0$ implies that

$$x(n) = \sum_{i=0}^{n-1} \rho^i w(n-i).$$

a.

$$\begin{aligned} E[x(n)] &= \sum_{i=0}^{n-1} \rho^i E[w(n-i)] \\ &= \sum_{i=0}^{n-1} \rho^i \cdot 0 = 0. \end{aligned}$$

b. Assume ρ is real. For $n > m$,

$$\begin{aligned} K_X(m,n) &= E[x(m)x(n)] \\ &= E\left[\sum_{i=0}^{m-1} \rho^i w(m-i) \sum_{j=0}^{n-1} \rho^j w(n-j)\right] \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \rho^{i+j} E[w(m-i)w(n-j)] \\ &= \sigma^2 \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \rho^{i+j} \delta_K(m-n-i+j). \end{aligned}$$

where

$$\delta_K(m-n-i+j) = \begin{cases} 1 & \text{if } j = n-m+i \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$K_X(m,n) = \sigma^2 \rho^{n-m} \sum_{i=0}^{m-1} \rho^{2i} = \sigma^2 \rho^{n-m} \frac{1-\rho^{2m}}{1-\rho^2}. \tag{1}$$

When $m > n$, $\delta_K(m-n-i+j) = 1$ when $i = m-n+j$, so

$$K_X(m,n) = \sigma^2 \rho^{m-n} \sum_{j=0}^{n-1} \rho^{2j} = \sigma^2 \rho^{m-n} \frac{1-\rho^{2n}}{1-\rho^2}. \tag{2}$$

Combining (1) and (2),

$$K_X(m,n) = \sigma^2 \rho^{|m-n|} \frac{1-\rho^{2\min(m,n)}}{1-\rho^2}.$$

c. When $|\rho| < 1$, as m and n get large,

$$K_X(m,n) \to \frac{\sigma^2 \rho^{|m-n|}}{1-\rho^2}.$$

and $x(n)$ is asymptotically stationary.