EE 562a

Homework 6

Due Wednesday July 15, 2009

Work the following 7 problems.

Problem 1. Let $X_1, \ldots X_n$ be n random variables each with mean μ and variance $\sigma^2 < \infty$. Let us estimate the variance as

$$\hat{\sigma^2} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

a. Show that $\hat{\sigma^2}$ is an unbiased estimator for σ^2.

b. Show that $\hat{\sigma^2} \to \sigma^2$ in probability.

Problem 2. Give an example of two random variables X and Y in which the best linear predictor of Y given X is a constant (has no predictive value) whereas the best predictor of Y given X predicts Y perfectly (without error).

Problem 3. For the following pairs of data find the best least squares fit for a model of the form

$$z_i = a + bx_i + \epsilon_i,$$

with $E[\epsilon_i] = 0$, $Var(\epsilon_i) = \sigma^2$, $E[\epsilon_i \epsilon_j] = 0$ for $i \neq j$, that is, find the regression line $a + bx$.

$$(x_i, z_i) =$$

$$(1.0, 4.5), (1.1, 5.3), (1.5, 6.2), (2.0, 6.4), (3.2, 9.0), (4.0, 10.1), (4.5, 11.5), (5.0, 13.3).$$

Problem 4. Consider the random process

$$X(u,t) = Be^{-A(u)t}$$

where $A(u)$ is a random variable with probability density function

$$f_A(x) = \begin{cases} ce^{-cx}, & c > 0,\ x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean, correlation and covariance functions of $X(u,t)$.

Problem 5. Find the mean and covariance functions for the random process

$$X(u,t) = A(u,t)\cos(2\pi f_0 t + \phi(u))$$

where $\phi(u)$ is a random variable uniformly distributed on the interval $(-\pi,\ \pi)$ and is independent of the amplitude modulation $A(u,t)$. Assume the correlation function, $R_A(t_1,t_2)$, of the modulation is known.

Problem 6. Consider the complex random process

$$X(u, t) = e^{i2\pi f(u)t}$$

where $f(u)$ is a uniform random variable on $\Omega = [f_0 - \Delta f, f_0 + \Delta f]$, i.e., the sample paths of the random process are complex sinusoids (or exponentials) with frequencies in the range $f_0 \pm \Delta f$. The index set T is the real line. Compute the mean and covariance functions of the random process.

Problem 7. Let $w(n)$ be an independent random sequence with mean zero and variance σ^2. Define the new random sequence $x(n)$ as

$$x(0) = 0,$$

$$x(n) = \rho x(n-1) + w(n), \quad n \geq 1.$$

a. Find the mean of $x(n)$ for $n \geq 0$.

b. Find the covariance of $x(n)$, denoted $K_X(m, n)$.

c. For what values of ρ does $K_X(m, n)$ converge to some finite valued function $g(m - n)$ as m and n become large (this is called *asymptotic stationarity*).