# EE 562a

**Homework 5**

Due Wednesday July 8, 2009

**Work the following 5 problems**.

**Problem 1.** Let $l_2 = l_2[0, \infty)$ be the space of all sequences $x = (x_1, x_2, \ldots)$ of complex numbers such that

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty.$$

Define

$$< x, y >= \sum_{i=1}^{\infty} x_i y_i^*.$$

Then $l_2$ is a Hilbert space.

Now let $H$ be any Hilbert space. Then $M \subseteq H$ is a called a *linear manifold* in $H$ if for all $x, y \in M$ and all complex numbers $\alpha$ we have $x + y \in M$ and $\alpha x \in M$. (Note: When $H$ is finite dimensional then a linear manifold is a vector subspace.)

Let

$$M = \{x \in l_2 : x = (x_1, x_2, \ldots) \text{ and all but finitely many } x_k \text{ equal zero}\}.$$

Show that $M$ is a linear manifold in $H$ that is not closed.

*Hint:* Construct a Cauchy sequence in $M$ that does not converge to an element in $M$.

**Solution.**

I. Show that $M$ is a linear manifold in $l_2$.
Clearly, every element $x \in M$ belongs to $l_2$ from the definition of $M$. Also for $x, y \in M$ and all complex numbers $\alpha$, $x + y \in l_2$ and $\alpha x \in l_2$ since $l_2$ is a linear space. The only thing we need to show is both $x + y$ and $\alpha x$ have finitely many non-zero components. Define $N(x)$ as the number of non-zero

components in $x$. Then clearly $N(\alpha x) = N(x)$ and $N(x+y) \leq N(x)+N(y)$. Since $N(x)$ and $N(y)$ are both finite, $N(\alpha x)$ and $N(x+y)$ are also finite. So, $x+y \in M$ and $\alpha x \in M$, hence $M$ is a linear manifold in $l_2$.

II. Show that $M$ is not closed.
Define the following sequence of sequences:

$$\begin{aligned} y_n &= (y_{n1}, y_{n2}, y_{n3}, \ldots, y_{nn}, 0, 0, \ldots) \\ &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, 0, 0, \ldots\right). \end{aligned}$$

Note $y_n$ has only fintiely $n$ non-zero components and:

$$\sum_{i=1}^{\infty} |y_{ni}|^2 = \sum_{i=1}^{n} (2^{-i})^2 < \sum_{i=1}^{\infty} (2^{-i})^2 = \frac{1}{3} < \infty.$$

So, $y_n \in M \forall n$. Also, for $m > n$,

$$\begin{aligned} \lim_{m,n\to\infty} |y_m - y_n|^2 &= \lim_{m,n\to\infty} \left(\sum_{i=n}^{m} \frac{1}{2^n}\right)^2 \\ &= \lim_{n\to\infty} \left(\sum_{i=n}^{\infty} \frac{1}{2^n}\right)^2 \\ &= \lim_{n\to\infty} \frac{1}{3} \times 4^{-(n-1)} = 0. \end{aligned}$$

So, $y_n$ is a Cauchy sequence in $M$. But clearly $y_n$ converges to $y = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right)$, which has infinite non-zero components, i.e, $y \notin M$.

So, $M$ is a linear manifold in $H$ that is not closed.

**Problem 2.** If $X$ and $Y$ are any two random variables, find the best estimator (or predictor) of $Y$ given $X$ for mean absolute error, i.e., find $g(X)$ so that $E[|Y - g(X)|]$ is minimal.

**Solution.**

Using the procedure shown in class,

$$E[|Y - g(X)|] = E_X \left[E_{Y|X} \left[|Y - g(X)||X\right]\right].$$

if we minimize $E_{Y|X}[|Y-g(X)||X]$ for all $X$, we minimize $E[|Y-g(X)|]$. Now,

$$E_{Y|X}[|Y-g(X)||X] = \int_{g(X)}^{\infty}(Y-g(X))f_{Y|X}(Y|X)dY + \int_{-\infty}^{g(X)}(g(X)-Y)f_{Y|X}(Y|X)dY. \tag{1}$$

Take derivative with respect to $g(X)$ using Leibnitz's rule:

$$\frac{\partial}{\partial u}\int_{\phi_1(u)}^{\phi_2(u)} h(u,v)dv = \int_{\phi_1(u)}^{\phi_2(u)} \frac{\partial h(u,v)}{\partial u}dv + \frac{d\phi_2(u)}{du}h(u,\phi_2(u)) - \frac{d\phi_1(u)}{du}h(u,\phi_1(u)). \tag{2}$$

For the first integral in (1), let $u = g(X)$, $v = Y$, $\phi_1(u) = g(X)$ and $\phi_2(u) = \infty$. Then the last 2 terms in (2) are 0 since $\frac{d\phi_2(u)}{du} = 0$ and $h(u,\phi_1(u)) = (g(X)-g(X))f_{Y|X}(Y|X) = 0$. Similar reasoning goes for the second integral in (1). Hence,

$$\frac{\partial}{\partial u}E_{Y|X}[|Y-g(X)||X] = \int_{-\infty}^{g(X)} f_{Y|X}(Y|X)dY - \int_{g(X)}^{\infty} f_{Y|X}(Y|X)dY.$$

So the optimal $g(X)$ should satisfy:

$$\int_{-\infty}^{g(X)} f_{Y|X}(Y|X)dY = \int_{g(X)}^{\infty} f_{Y|X}(Y|X)dY.$$

That is, $g(X)$ is the median of the posterior PDF.

**Problem 3.** Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi Ft + \theta).$$

Here the amplitude $A$ is a constant but we do *not* know its value. We do know that the frequency $F$ is 200 Hz and the phase $\theta$ is $\pi/4$. We can follow the steps below to estimate the value of $A$. Assume for purposes of calculation that the value of $A$ is 10, i.e., use $A = 10$ in the above signal in your calculations.

**S1**. Multiply $r_a(t)$ by $x(t)$, where $x(t) = \cos(2\pi 200t + \pi/4)$. Call the result $y(t)$.

**S2**. Integrate $y(t)$ from 0 to $T$ and multiply the result by $2/T$. The result is your estimate of $A$.

a. Follow the 2 steps above and estimate $A$ using $T = 3,\ 7,\ 17,\ 27,\ 127$ msec.

b. Explain why following the 2 steps above will give the exact answer for $A$ as $T \to \infty$.

c. Determine (analytically) the *finite* values of $T$ that will make your estimate for $A$ exact and using the smallest such $T$ follow the two steps above again to estimate $A$.

**Solution.**

a. Following the 2 steps above, we get

| T(msec) | $\hat{A}$ |
|---|---|
| 3 | 9.0836 |
| 7 | 9.6072 |
| 17 | 9.8383 |
| 27 | 9.8982 |
| 127 | 9.9784 |

b. Expanding the expression for $y(t)$:

$$\begin{aligned} y(t) &= A\cos(2\pi F t + \theta)\cos(2\pi 200 t + \frac{\pi}{4}) \\ &= \frac{A}{2}[\cos(2\pi(F+200)t + (\theta + \frac{\pi}{4})) + \cos(2\pi(F-200)t + (\theta - \frac{\pi}{4}))] \\ &= \frac{A}{2}[\cos(2\pi 400 t + \frac{\pi}{2}) + 1], \end{aligned}$$

since $F = 200$ and $\theta = \frac{\pi}{4}$. Hence the estimate of A can be derived as,

$$\begin{aligned} \hat{A} &= \frac{2}{T}\int_0^T \frac{A}{2}\cos(2\pi 400 t + \frac{\pi}{2})dt + \frac{2}{T}\int_0^T \frac{A}{2}dt \\ &= A + \frac{2}{T}\int_0^T \frac{A}{2}\cos(2\pi 400 t + \frac{\pi}{2})dt. \end{aligned} \tag{3}$$

Since the cosine function is bounded between $\{-1, +1\}$, the second term goes to zero as $T \to \infty$.

c. From (3), we see that if $2\pi 400T = 2\pi K$ for integer $K$, the second term will always be zero. Hence, to make the estimate for $A$ exact, $T = K/400$ sec. The smallest such $T = 2.5$ msec. Using $T = 2.5$ msec, we calculate $\hat{A} = 10$.

**Problem 4.** Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi 200t + \theta)$$

and sample it at 500 Hz to get the digital signal

$$r(n) = A\cos(0.8\pi n + \theta).$$

Here the amplitude $A$ is a constant but we do *not* know its value. Furthermore, we do *not* know the $\theta$ phase value. We can follow the steps below to estimate the value of $A$. Assume for purposes of calculation that the value of $A$ is 10 and $\theta = \pi/4$, i.e., use $A = 10$ and $\theta = \pi/4$ in the above signal in your calculations. You can work this problem using Matlab or Excel or some other software except for the analytic part.

**S1**. Multiply $r(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.8\pi n)$ and $x_2(n) = \sin(0.8\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

**S2**. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some $N$) and take the average of each (divide by $N$) and then multiply the averages by 2. Call the results $z_1$ and $z_2$, respectively.

**S3**. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of $A$.

a. Follow the 3 steps above and estimate $A$ using $N = 5, 11, 19, 29$.

b. Explain why following the 3 steps above will give the exact answer for $A$ as $N \to \infty$.

d. Determine (analytically) the *finite* values of $N$ that will make your estimate for $A$ exact and using the smallest such $N$ follow the 3 steps above again to estimate $A$.

**Solution.**

a. Following the 3 steps above, we get

| $N$ | $\hat{A}$ |
|---|---|
| 5 | 10.0000 |
| 11 | 10.0412 |
| 19 | 10.5018 |
| 29 | 10.3285 |

b. Expanding the terms,

$$\begin{aligned} y_1(n) &= \frac{A}{2}[\cos(1.6\pi n + \theta) + \cos(\theta)] \\ y_2(n) &= \frac{A}{2}[\sin(1.6\pi n + \theta) - \sin(\theta)] \end{aligned}$$

$$\begin{aligned} z_1 &= A\cos(\theta) + \frac{A}{N}\sum_{n=0}^{N-1}\cos(1.6\pi n + \theta) \\ z_2 &= -A\sin(\theta) + \frac{A}{N}\sum_{n=0}^{N-1}\sin(1.6\pi n + \theta) \end{aligned}$$

Define $\Sigma_c \triangleq \sum_{n=0}^{N-1}\cos(1.6\pi n + \theta)$ and $\Sigma_s \triangleq \sum_{n=0}^{N-1}\sin(1.6\pi n + \theta)$, we have

$$\begin{aligned} z_1^2 + z_2^2 &= \left(\frac{A}{N}\Sigma_c + A\cos(\theta)\right)^2 + \left(\frac{A}{N}\Sigma_s + A\sin(\theta)\right)^2 \\ &= A^2 + \left(\frac{A}{N}\Sigma_c\right)^2 + \left(\frac{A}{N}\Sigma_s\right)^2 + \frac{2A^2}{N}\left[\cos\theta\Sigma_c - \sin\theta\Sigma_s\right]. \end{aligned}$$

For $N \to \infty$, all but the first term go to zero. The estimate of A is thus,

$$\hat{A} = \sqrt{z_1^2 + z_2^2} \;=\; A.$$

c. We see that the terms $\Sigma_c$ and $\Sigma_s$ are always zero if we sum over full cycles of the sine and cosine terms. Since $N$ has to be integer, we choose $N = K\frac{LCM(1.6,2)}{1.6}$ for integer $K$, where LCM refers to the Lowest Common Multiple. Using the smallest such $N = 5$, we calculate $\hat{A} = 10$.

**Problem 5.** Suppose we receive the quantized digital signal

$$r_q(n) = \text{Round}\left[A\cos(0.8\pi n + \theta)\right]$$

where 'Round' means the samples are rounded to the nearest integer (in a real analog to digital (A/D) converter this rounding would map to a certain level of the A/D depending on the number of levels of the A/D but it is sufficient for this problem to just assume the rounding results in the integer obtained by the usual rounding operation). The amplitude $A$ is a constant but we do not know its value. Furthermore, we do not know that the phase is $\theta = \pi/8$. We can follow the steps below to estimate the value of $A$. [For purposes of calculation let $A$ actually have the value 5.]

**S1**. Multiply $r_q(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.80\pi n)$ and $x_2(n) = \sin(0.80\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

**S2**. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some $N$) and take the average of each (divide by $N$) and then multiply the averages by 2. Call the results $z_1$ and $z_2$, respectively.

**S3**. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of $A$.

a. Follow the 3 steps above and estimate $A$ using $N = 5$.

b. Repeat (a) for $N = 10$.

c. Making $N$ multiples of 5 estimates $A$ exactly if the input samples were not quantized. Based on your answers to parts (a) and (b) what would your estimate for $A$ be if $N$ is 5000. Explain why the estimate is not becoming exact even for very large $N$. If you were designing the system what could you do to avoid this problem?

**Solution.**

a. Following the 3 steps above, we get $\hat{A} = 5.2149$ for $N = 5$.

b. Repeating (a) for $N = 10$, we get $\hat{A} = 5.2149$.

c. Based on the answers to parts (a) and (b), the estimate for $A$ would be $\hat{A} = 5.2149$ if $N$ is 5000.

The quantized system can be expressed as

$$r_q(n) = r_a(n) + e(n).$$

where $r_a(n)$ is the exact sampled signal and $e(n)$ is the quantization error. Then, we can expand the terms as,

$$\begin{aligned} z_1 &= \frac{2}{N}\sum_{n=0}^{N-1} r_a(n)x_1(n) + \frac{2}{N}\sum_{n=0}^{N-1} e(n)x_1(n) \\ z_2 &= \frac{2}{N}\sum_{n=0}^{N-1} r_a(n)x_2(n) + \frac{2}{N}\sum_{n=0}^{N-1} e(n)x_2(n) \end{aligned}$$

Using the result from Q4, we see that for $N =$ integer multiples of 5,

$$\begin{aligned} \frac{2}{N}\sum_{n=0}^{N-1} r_a(n)x_1(n) &= A\cos\theta \\ \frac{2}{N}\sum_{n=0}^{N-1} r_a(n)x_2(n) &= -A\sin\theta \end{aligned}$$

Define $\epsilon_1(n) \triangleq \sum_{n=0}^{N-1} e(n)x_1(n)$ and $\epsilon_2(n) \triangleq \sum_{n=0}^{N-1} e(n)x_2(n)$, the estimate of $A$ can be expressed as,

$$\hat{A} = \sqrt{A^2 + \epsilon_2^2 + \epsilon_2^2 + 2\epsilon_1 A\cos\theta - 2\epsilon_2 A\sin\theta}.$$

Due to the non-zero error terms, we cannot get $\hat{A} = A$, even when $N =$ integer multiples of 5.

Since we are restricted to quantizing at integer values, to mitigate the error, we can amplify the signal before quantization to increase the quantization levels and reduce the effective error $e(n)$. We then compensate $\hat{A}$ by the same factor after estimation. For example, an amplification factor of 10 results in $\hat{A} = 4.9930$ for $N = 5$.